

Online Appendix to Fair Payments for Efficient Allocations in Public Sector Combinatorial Auctions

Robert W. Day
Operations and Information Management
School of Business
University of Connecticut
Storrs, CT 06269-1041

S. Raghavan
Decision and Information Technologies
The Robert H. Smith School of Business
University of Maryland
College Park, MD 20742-1815

Online Appendix

Proof of Theorem 3.1

First, we prove the initial statement, that any bidder-Pareto-optimal core point corresponds to a Nash Equilibrium in semi-sincere strategies when using a mechanism that is bidder-Pareto-optimal with respect to submitted preferences. To construct a proof by contradiction, suppose that for any specific point in the core (π^A, A) , all bidders bidding $b_j(\cdot) = \max(v_j(\cdot) - v_j(S_j^A) + \pi_j^A, 0)$ is not a Nash equilibrium. Thus there is a bidder l who may profitably deviate from this bid profile (unilaterally), achieving a new winning coalition D with payment vector π^D , and allocating bundle S_j^D to each bidder j . It may be worth noting that l must have been a winner in the allocation A . (If not, l must bid more than b_l on some bundle, which implicitly equals v_l , since when losing $S_l^A = \emptyset$ and $\pi_l = 0$. But when this losing bidder increases a bundle bid to the minimum amount necessary to be included in an efficiently determined allocation, she will pay her bid, which would be greater than her value, contradicting the profitability of the deviation.)

The profitability of the new deviation strategy for bidder l implies that $v_l(S_l^A) - \pi_l^A < v_l(S_l^D) - \pi_l^D$, or equivalently, $v_l(S_l^D) - v_l(S_l^A) + \pi_l^A > \pi_l^D$. Because the left side of this inequality is exactly the bid of l prior to deviation we may write this as follows:

Observation 1: $b_l(S_l^D) > \pi_l^D$

Next, note that because of bidder-Pareto-optimality in the core of the mechanism (with respect to submitted bids) and the bid profiles $b_j(\cdot)$, for any winning bidder j there is a coalition not including j whose maximum total bids on the entire set of auction items is equal to the payments of the winners. If this were not true, the mechanism could find a Pareto improvement, lowering the payment of j without disturbing the surplus of other players and without leaving the (apparent) core. In particular, for the initial allocation to the coalition A , we note the following:

Observation 2: There is a coalition B such that $l \notin B$ and $\sum_{j \in B} b_j(S_j^B) = \sum_{j \in A} \pi_j^A$.

Thirdly, we note that under the initial strategies (prior to deviation by l) we have $\sum_{j \in A} \pi_j^A \geq \sum_{j \in D} (v_j(S_j^D) - v_j(S_j^A) + \pi_j^A) = \sum_{j \in D} b_j(S_j^D)$, with the first inequality following because (π^A, A) is in the core with respect to true valuations. We re-state this as the following:

Observation 3: $\sum_{j \in A} \pi_j^A \geq \sum_{j \in D} b_j(S_j^D)$

Putting these facts together, and considering what bidders in B are willing to offer the auctioneer after

the deviation of bidder l according to their own undeviated bids, we have the following string of inequalities:

$$\begin{aligned}
\sum_{j \in B} (b_j(S_j^B) - b_j(S_j^D) + \pi_j^D) &= \sum_{j \in A} \pi_j^A + \sum_{j \in B} (\pi_j^D - b_j(S_j^D)) && \text{by Observation 2 and the selection of } B \\
&\geq \sum_{j \in D} b_j(S_j^D) + \sum_{j \in B} (\pi_j^D - b_j(S_j^D)) && \text{by Observation 3} \\
&> \pi_l^D + \sum_{j \in D \setminus l} b_j(S_j^D) + \sum_{j \in B} (\pi_j^D - b_j(S_j^D)) && \text{by Observation 1} \\
&= \sum_{j \in D} \pi_j^D + \sum_{j \in B \setminus D} (\pi_j^D - b_j(S_j^D)) && \text{by canceling terms in } B \cap D \\
&= \sum_{j \in D} \pi_j^D
\end{aligned}$$

because each $S_j^D = \emptyset$ for bidders not in D , and thus $\pi_j^D = b_j(S_j^D) = 0$ for all bidders j in $B \setminus D$. Altogether this implies that:

$$\sum_{j \in B} (b_j(S_j^B) - b_j(S_j^D) + \pi_j^D) > \sum_{j \in D} \pi_j^D$$

But with Lemma 4.1, this implies that the coalition B would be willing to pay strictly more for the bundles S_j^B following the deviation by bidder l than coalition D actually pays following the deviation, contradicting the core property of the mechanism based on the submitted bids. Thus, every bidder-Pareto-optimal core point has a semi-sincere bid profile supporting that point as a Nash equilibrium.

To prove the second claim in the statement of the theorem, suppose that we have strategies $b_j(\cdot)$ for each j that together form a Nash equilibrium with winning coalition A who pay π_j^A and each receive bundle S_j^A , and that this outcome is not in the core with respect to true valuations. That is, there is some coalition B that objects to the the payments π_j^A :

$$\sum_{j \in A} \pi_j^A < \sum_{j \in B} (v_j(S_j^B) - v_j(S_j^A) + \pi_j^A)$$

even though

$$\sum_{j \in A} \pi_j^A \geq \sum_{j \in B} (b_j(S_j^B) - b_j(S_j^A) + \pi_j^A)$$

by the core property of the mechanism with respect to submitted bids. Rearranging we have $\sum_{j \in B} (v_j(S_j^B) - b_j(S_j^B)) > \sum_{j \in B} (v_j(S_j^A) - b_j(S_j^B))$ and thus $v_l(S_l^B) - b_l(S_l^B) > v_l(S_l^A) - b_l(S_l^A)$ for at least one bidder $l \in B$. But this says that in a Nash equilibrium bidder l shades (reduces her bid) more on bundle S_l^B than on bundle S_l^A , while Theorem 3a of Ausubel and Milgrom (2002) shows that a semi-sincere strategy with equal shading on all bundles is always a best response for every bidder when the mechanism is bidder-Pareto-optimal in the core with respect to the submitted bids, a contradiction. (Theorem 3a of Ausubel and Milgrom (2002))

is stated in terms of the ascending proxy auction, but since the only property of that auction used in the proof is bidder-Pareto-optimality in the core with respect to the submitted bids, the result immediately generalizes.) Thus every Nash equilibrium profile of bids results in a core outcome with respect to true valuations.

Finally, we must show that every Nash equilibrium in semi-sincere strategies is bidder-Pareto-optimal within the core. If a collection of π_j^A s and S_j^A s represent a Nash equilibrium that is not bidder-Pareto-optimal in the core, then there is some alternative bidder-Pareto-optimal outcome in the core with π_j^B and S_j^B for each bidder and at least one bidder l that prefers outcome B , while every other bidder is at worst indifferent. By deviating to $b_l(\cdot) = \max(v_l(\cdot) - v_l(S_l^B) + \pi_l^B, 0)$ bidder l can obtain utility of $v_l(S_l^B) - \pi_l^B$ since no other outcome is bidder-Pareto-optimal in the core with respect to the preferences reported following the deviation. By our selection of l this deviation obtains positive gains, contradicting our assumption of a Nash equilibrium.

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