ABSTRACT

Title of dissertation:	EXPRESSING PREFERENCES WITH PRICE-VECTOR AGENTS IN COMBINATORIAL AUCTIONS
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In this work, we investigate two combinatorial auction formats in which each bidding bidder may be represented by a collection of unit-demand or price-vector agents. In the first model, bidder preferences are aggregated in a *Bid Table*, through which a bidder in a combinatorial auction may express several forms of subadditive preferences. We show that the gross substitutes property holds for this model, and design a large-scale combinatorial auction using bid tables as a demand revelation stage, determining linear price signals for later stages. The constraints of

this model coincide naturally with the restrictions of recently proposed FAA landing-slot auctions, and we provide a slot auction design based on this model.

In a second model, we explore the more complex behavior possible when each bidder's collection of price-vector agents coordinate based on a bidder-specified ordering of the auction items. With this coordination each bidder is able to convey a rich set of preferences, including the ability to express both superadditive and subadditive bundle synergies (i.e., substitutes and complements). The instructions for this collection of agents are tabulated in a lower-triangular *Matrix Bid*, and we compare the use of matrix bids to other compact techniques for writing down a wide variety of bidding information. We show that the winner determination problem for this Matrix Bid Auction is NP-hard, provide results from a series of computational experiments, and develop IP techniques for improving run time.

In addition to the results on price-vector agents, bid tables, and matrix bidding, we present a new technique for achieving bidder-Pareto-optimal core outcomes in a sealed-bid combinatorial auction. The key idea of this iterative procedure is the formulation of the separation problem for core constraints at an arbitrary point in winner payment space.

EXPRESSING PREFERENCES WITH PRICE-VECTOR AGENTS IN COMBINATORIAL AUCTIONS

by

Robert W. Day

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Advisory Committee:

Professor Subramanian Raghavan, Chair
Professor Michael Ball
Professor Peter Cramton
Professor Bruce Golden
Professor Dianne O'Leary

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Contents

LIST OF FIGURES

LIST OF ABBREVIATIONS

i	index for a particular item
j	index for a particular bidder
k	index for a particular agent
N	number of unique items in an auction
M	number of bidders in an auction
A_j	number of agents belonging to bidder j
$I = \{1, 2, ...i, ...N\}$	set of items being auctioned
$J = \{1, 2, ...j, ...M\}$	set of bidders participating in an auction
$K_j = \{1, 2, ...k, ...A_j\}$	set of agents belonging to bidder j
$v_j(S)$	value of itemset S to bidder j
v_{ijk}	value of item i if received by bidder j's kth agent
$b_j(S)$	bid on itemset S by bidder j
b_{ijk}	bid on item i if received by bidder j's kth agent
$u_j(S, p)$	utility to bidder j receiving itemset S at price vector p
$D_j(p)$	set of all bundles demanded by bidder j at price vector p

$i \rightarrow j, k$	item i is awarded to bidder j's kth agent in the selected efficient allocation
π_j	payment made by bidder j
W	the set of winning bidders
π^t	a vector of payments for $j \in W$ at iteration t
$z_C(\pi^t)$	the coalitional value for C relative to payments π^t
MBA	a Matrix Bid Auction
$*$	a significantly large negative number

CHAPTER 1

Introduction

Auctions have long been used as a mechanism to allocate and determine a price or competitive value for scarce resources. In the familiar *English auction*, an auctioneer calls out prices on a single item or lot while bidders signal their willingness to pay the current price until a single bidder remains, the winner of the item. This mechanism arrives at an outcome that is *efficient* (the item for sale is given to the bidder who values it the most) and the payment made is just enough to beat the closest competitor. In addition, this simple auction has the desirable properties of *demand-revelation* (the bidders get to learn about the preferences of other bidders over the course of the auction), *privacy-preservation* (the winner never tells the highest amount she is willing to pay) and *incentive-compatibility* (the best strategy for each bidder is honest revelation; she stays in the auction until her true valuation is met.) Krishna [**34**], for example, outlines these and other basic properties of single-item auctions.

The situation becomes more difficult when the seller wishes to sell several items at once. A bidder may perceive some items to be complements and others to be substitutes and will face difficulties participating in several English auctions (or other single-item auctions). For example, if a bidder perceives some collection of items to be substitutes then it is difficult to know which single-item auction to compete in to win just one item. On the other hand, when the items are complements a series of single-item auctions forces the bidder to face an *exposure problem*; a bidder must win an auction for the first item without knowing whether or not she will be able to win the auction for the second item, and so forth. The presence of substitutabilities and complementarities (which we will collectively call *synergies*) make it impossible to guarantee an efficient auction outcome from a series of single-item auctions. This difficulty has motivated the development of *combinatorial auctions* in which items are awarded and bidder preferences are expressed over combinations or sets of items.

Given a set $I = \{1, 2, ...i, ...N\}$ of N unique indivisible items, each bidder $j \in J = \{1, 2, ...j, ...M\}$ in a general combinatorial auction can be modeled by a value function $v_j : 2^I \rightarrow \mathbb{R}$ and a bidding function $b_j : 2^I \rightarrow \mathbb{R}$. To make precise the notion of synergy, the primary motivation for the use of combinatorial auctions, we adopt the following definition of the synergy perceived by bidder j on set $S \subseteq I$:

$$\sigma_j(S) = v_j(S) - \sum_{i \in S} v_j(\{i\})$$

If synergy is positive, the bundle contains complements, if negative, mostly substitutes (as perceived by bidder j). As an adjectival alternative, we may refer to preferences as subadditive, additive, or superadditive when bundle preferences experience negative, zero, or positive synergy, respectively.

This definition of synergy emphasizes the importance of both positive and negative synergy which should both be considered in the design of a combinatorial auction. If negative synergy is ignored (because, perhaps, the immediate benefits to the auctioneer are not as obvious), bidders will feel the need to reduce their bids because of exposure to the possibility of receiving substitute items at too high a price, potentially leading to lost auction revenue. The presence of nonzero synergy motivates an auction format in which each bidder discloses some bidding function b_j over bundles, and the auctioneer solves a combinatorial optimization problem to find an allocation maximizing total bid revenue. This general *winner-determination* problem can be formulated as an Integer Program (IP), with binary variables $x_j(S)$ that equal 1 if and only if bidder j is awarded bundle $S \subseteq I$:

$$\text{(GWD)} \qquad \max \sum_{j \in J} \sum_{S \subseteq I} b_j(S) \cdot x_j(S)$$

$$\text{(1.1)} \qquad \text{subject to} \quad \sum_{S \supseteq \{i\}} \sum_{j \in J} x_j(S) \leq 1, \qquad \forall i \in I$$

$$\text{(1.2)} \qquad \sum_{S \subseteq I} x_j(S) \leq 1, \qquad \forall j \in J$$

$$\text{(1.3)} \qquad x_j(S) \in \{0, 1\}, \qquad \forall S \subseteq I, \forall j \in J$$

Constraints (1.1) ensure that each item is assigned to at most one bidder, while constraint set (1.2) prevents the auctioneer from accepting multiple bids from the same bidder.

It is well known in auction theory that a set of Vickrey-Clark-Groves (VCG) prices may be computed that ensure each bidder will find it in her best interest to bid truthfully, a property known as incentive compatibility [**68**][**13**][**25**]. This is accomplished by computing a discount for each winning bidder by re-solving GWD with that bidder removed. The VCG discount for bidder j is equal to the difference between the objective value of GWD with all bidders and the objective value of GWD with bidder j removed. If payments are assigned to winning bundles so as to ensure truthful-revelation (i.e., if the VCG prices are used) then bidders will submit bids $b_j = v_j \ \forall j \in J$ and the optimal outcome to GWD is efficient.

Despite its beauty and simplicity, this approach has not been widely implemented for several reasons. VCG mechanisms are susceptible to collusion and false-name (or shill) bidding and do not necessarily generate enough revenue for the outcome to be considered stable (in the sense of a core equilibrium). The general winner determination problem (as formulated in GWD or otherwise) is $\mathcal{NP}$-hard; thus we cannot guarantee speedy (i.e., polynomial-time) convergence of an algorithm for its solution except for very small problems. Further, the mechanism for preference-revelation is direct: each bidder directly specifies a bid on every bundle.

This approach does not allow for demand-revelation; a bidder must write down prices for all bundles without knowing what prices her competitor's will name, despite the fact the true value a bidder is willing to pay may be dependent on such information. Privacy preservation is also compromised; the auctioneer knows the true valuations of the winning bidders even though they pay less than the full amount. Finally, the direct revelation of preferences may be exhausting for the bidder; evaluating each of the $2^N - 1$ nonempty bundles of items will be impractical for more than just a few items.

Several approaches have been suggested in the literature for conducting a combinatorial auction that regains some of the desirable properties of the English auction that are lost in a sealed-bid VCG implementation of GWD. We discuss a number of these approaches and their relationship to our own work in Chapter 2. Given the exponential data requirement to express preferences on every bundle and the arbitrary nature of the restricted subset approach (discussed in §2.2.4) there is no consensus in the literature on a single best combinatorial auction structure. Instead, we pursue specialized algorithms and mechanisms tailored to specific markets or bidder profiles.

In this dissertation, we concentrate on the *exponential bundles problem* of combinatorial auctions and investigate new compact methods for a bidder to write down bid information as an alternative to assigning a price to every bundle explicitly. Both of the approaches for preference elicitation explored here make use of the concept of a *price-vector agent*. Each agent can be described as a fictional entity representing some portion of the preferences of a particular bidder. A price-vector agent is assigned a vector of prices (a monetary amount for each of the items in the auction) and “participates” in the auction based on these prices. An agent receiving a particular item pays at most the price vector component associated with that item. Throughout we assume that each price-vector agent is a unit-demand agent; each agent may receive at most one item. This clarifies the role of the price-vector agent in economic terms: Each agent treats all items as perfect substitutes. Expression of preferences using price-vector agents encourages each bidder to decompose his preferences into collections of substitutable items and to dedicate one or several agents to this collection depending on the level of substitutability (pure versus partial).

With no further structure or rules of coordination among the agents, we call the collected set of (column) price-vectors a *bid table*. The result is any easy to read method of compactly annotating certain forms of substitutable preferences. Figure 1.1 shows an example of a bid table and demonstrates how a single agent can be used to name prices for pure substitutes, or how a collection of agents can combine to yield a decreasing offer on partially substitutable items. To interpret a bid table, one need only keep in mind that at most one bid entry may be accepted from each

Figure 1.1. A Bid Table

	agent 1	*agent* 2	*agent* 3	*agent* 4
pure substitute A_1	15	0	0	0
pure substitute A_2	16	0	0	0
pure substitute A_3	17	0	0	0
partial substitute B_1	0	23	18	12
partial substitute B_2	0	20	16	10
partial substitute B_3	0	19	15	9

row, and at most one from each column. We see therefore that in the example of Figure 1.1 that items A_1, A_2, and A_3 are indeed pure substitutes, because at most one can be purchased at a positive price. Similarly, if the bidder of this example receives item B_1 priced at 23, she cannot also receive (for example) item B_2 at a price of 20. If item B_2 is assigned to the bidder, the revenue maximizing auctioneer would be forced to accept a lower price from another agent of that bidder. In this case the auctioneer would charge 16 for B_2 rather than 20, verifying the partial substitutability; B_2 is worth less if taken with B_1.

In Chapter 3, we show that the set of preferences expressible in bid tables are properly contained in the set of preferences satisfying the *gross substitutes property*, connecting this theory to the economic literature on restricted preferences. We may then apply a theoretical result of Ausubel and Milgrom to elucidate a strength of the Bid Table Auction: a VCG mechanism may be used with no possibility of disruption from false-name bidding or joint deviation by losing bidders, properties not satisfied for a VCG mechanism in general. Despite this strength and the connection to applications in which bidders determine bundle value via optimization over an assignment network, the limitations presented by the gross substitutes property are indeed great. In particular, no superadditive or complementary preferences may be expressed.

The dissertation next divides in two directions, each concerning a different method for the expression of preferences for complementary bundles in the price-vector agent context. In Chapter 4, we show how a Bid Table Auction may be incorporated as a demand revelation phase in a three-stage hybrid auction. We demonstrate how supporting linear prices are extracted from the linear programming (LP) dual of the winner determination optimization, and how they may be used as signals of competitive activity and as eligibility requirements. The use of bid tables for initial demand revelation allows for both ease of submission and for reduced bid exposure relative to a Simultaneous Ascending Auction (SAA) filling the same role in a hybrid auction. The reduction of exposure allows a new entrant to simultaneously compete in several markets without the danger of exceeding her capacity for items. We develop our hybrid auction incorporating bid tables in the context of recently proposed Federal Aviation Administration (FAA) landing-slot auctions. The comprehensive design is summarized as follows: a linear price demand-revelation stage using bid tables, a demand-revelation stage with bundle prices, and a sealed-bid final stage achieving efficiency (relative to the submissions) and bidder-Pareto-optimal core prices.

The computation of bidder-Pareto-optimal core prices via the separation and generation of core constraints is treated independently in Chapter 5, due to its applicability in other sealed-bid auction contexts. Among this material we consider the *selection* of a bidder-Pareto-optimal core outcome among all Pareto efficient core points, a matter that is currently unsettled in the literature. In particular, we argue in favor of a core outcome that minimizes the total payments made by all bidders and show that the minimax techniques explored by Parkes [**54**] do not necessarily achieve this outcome. In addition, we show how a simple perturbation technique allows us to choose the most equitable outcome when there are several that minimize total payments.

In Chapter 6 and Chapter 7, we explore the use of price-vector agents with additional constraints applied to enforce a simple form of cooperation among the agents belonging to an individual bidder. The bidder in this context submits a ranking or strict ordering of the items in the auction, while the previously permutable price-vector agents are each given a defined rank. Each price loaded into an agent then represents what that agent is willing to pay for a particular item, given that the superior agent (the one coming just before this agent in the ranking) receives a superior item (any coming earlier in the ranking). We show that this "pecking

Figure 1.2. A Matrix Bid

	agent 1	*agent* 2	*agent* 3	*agent* 4	*agent* 5
item C	29				
item A	27	27			
item E	20	20	25		
item B	18	18	23	15	
item D	17	17	22	14	0

order" for receiving items among the agents provides enough structure to support a wide array of preference expression, including both superadditive and subadditive preferences.

Since the kth ranked agent can not receive an item unless each of the $k-1$ more highly ranked agents also receives items, it is unnecessary for that agent to price items with a rank superior to its own rank k. This generates a lower-triangular *matrix bid* for writing down preferences compactly, as demonstrated in Figure 1.2. In all matrix bids the first column contains the value for each item if received on its own (in the first column). In this example, the bidder perceives no synergy if awarded a second item (second column entries are identical to the first) but indicates that a willingness to pay 5 extra monetary units when a third item is received, reflecting (for example) an operational advantage of having at least three of the items. Similarly, the fourth best item received will be worth 3 units less than if awarded on its own, perhaps reflecting an additional expense to rent space for a fourth item. The entry of zero in the fifth column indicates that five items is beyond this bidder's capacity to consume; no additional payment will be given for a fifth item.

This example gives just a glimpse of the expressability possible with a single matrix bid. In Chapter 6 we explore the basic properties of matrix bidding and provide examples of several types of possible expressions. Motivated by other logically structured "bid languages" presented in the literature, we show that an individual matrix bid has the ability to contain any of the proposed "atomic" statements of preference used to build compound sentences of preference. Further, several of these bid atoms may be contained in a single matrix bid, which may also be supplemented with "salvage value" bids on any or all items. This allows the bidder to make a positive incremental offer on any item as an add-on to a specified package bid (of any type) whenever the additional items will bring positive value, and provides the ability to do so within the same atomic statement of preference. We show that a logical bid language with matrix bids as atoms is as expressive as the most robust logical language presented in the literature [**9**], and relative to that language may often require less atoms and less symbols to represent a given set of preferences.

We show that the winner-determination problem for a matrix bid auction is $\mathcal{NP}$-hard. In Chapter 7 we present computational findings on our ability to solve instances of this problem, including instances with as many as 50 items and 100 matrix bids. In this development we uncover some interesting properties and develop heuristics based on our experience dealing with fractional solutions to the LP relaxation of the IP winner determination problem. Among these techniques are the recognition of a class of cover-like inequalities with a heuristic for the corresponding separation problem, and an objective perturbation technique designed to favor integral extreme points over fractional solutions with the same objective value (an occurrence which was not rare in our initial computations).

Before describing our own results in greater depth, we first present a review of recent auction literature in Chapter 2, paying particular attention to research with an Operations Research (OR) flavor.

CHAPTER 2

Auction Literature

Research on auctions has been developed in the Economics literature through the study of strategic games [**68**] and in the OR literature as constrained optimization since the mid 20th century [**24**]. More recently, there has been a great deal of intellectual cross-pollination from Management Science, Information Systems, and Computer Science in an attempt to develop new electronic markets providing better economic outcomes through centralized computational decision making. The computational difficulties presented by these markets are not readily untied from their economic context, making it difficult to apply a purely algorithmic approach which ignores strategic considerations. Conversely, the most general economic allocation problems exhibit a level of computational complexity ($\mathcal{NP}$-hardness) that requires algorithmic expertise, making it difficult to attack these market design problems from a purely economic standpoint. This is especially true if we consider a very general definition of an auction:

Definition: An *auction* is a mechanism of information submission, together with rules for assigning items and payments to participants based on this submitted information.

Since the OR discipline emphasizes the successful implementation of decision making technology in real-world situations, we begin with a quick survey of the successful and growing applications of auction theory, keeping this general definition in mind. The most widely recognized electronic auction is undoubtedly eBay and similar electronic venues for commerce that may be described as mostly consumer-to-consumer. The success of these markets demonstrates that the "electronic" services (a web-based platform) provide added-value even when the "auction" structure is simple (one-seller, one-item, simple bidding rules). Though several interesting avenues of research receive attention from these consumer-to-consumer settings, the success and future of the OR approach to auctions seems to stem from the use of auctions in business-to-business (B2B) commerce and government allocations, where the value-added is derived from the ability of the auction mechanism itself to elicit and process information, determining a more favorable outcome in a centralized market. These auctions typically have a "combinatorial" component, in which bidders may express how their preferences vary over combinations or bundles of items. This quality at once promises to extract value from the auction decision mechanism, and places the computational challenge for the mechanism at the very frontier of what is possible through computational technique.

2.1. Auction Applications

2.1.1. Auctions and Governmental Allocation

From the perspective of applications, auction mechanisms first entered the OR literature in the context of government controlled allocation problems. Throughout this literature, auctions are proposed as a tool of deregulation. Rather than allowing governmentally controlled resources to be distributed via political decision-making criteria and regulatory control (typically an inefficient procedure), auctions provide a market-based mechanism to allocate government property fairly. Before any such paradigm shift can take place, researchers must first provide scientific evidence that the auction mechanism will produce favorable outcomes, and that the auction can be implemented smoothly and at a comfortable pace for all participants. The pursuit of such evidence for governmental problems inaugurated the field as it is known today, with the healthy mix of theory and implementation indicative of Management Science and OR.

Rassenti, Smith, and Bulfin [**59**] pioneered this line of work in 1982 with one of the first efforts to confront multi-item auction winner-determination problems with the computational techniques of Integer Programming (IP). In this work they propose an auction mechanism for the allocation of airport time-slots and give some of the first experimental verification (using test subjects) that the combinatorial auction paradigm can achieve more efficient outcomes than a non-combinatorial mechanism. The auctions and corresponding winner-determination IPs are modest in size, reflecting the difficulty and lack of scalability in solving these hard allocation problems.

Ironically, though the airport time-slot application was one of the first proposed for the use of combinatorial auctions, the FAA is only now beginning to consider the use of auctions to control demand for landing/take-off slots. Ball, Donohue, and Hoffman [**6**] provide a description of the necessary and desirable features in airport slot auction and provide the motivation for the airport landing-slot auction that we develop in Chapter 4. A parallel and quite thorough investigation of the pros-and-cons of airport-slot auctions in the European market is given by the consulting firm dotEcon [**21**]. Both emphasize the economic benefits of an auction mechanism for the allocation of landing-slots at congested airports, and advocate a combinatorial auction implementation to achieve the desired results. Most recently, NEXTOR (the National Center of Excellence for Aviation Operations Research) and the FAA have sponsored two workshops ([**48**], [**49**]) exploring the use of auctions for congestion control.

Another early influential work rooted in governmental applications is given by the Adaptive User Selection Mechanism (AUSM) of Banks, Ledyard, and Porter [**7**], proposed to allocate resources such as jobs on a super-computer or mission time aboard a space station. Like Rassenti, Smith, and Bulfin [**59**], this paper began to stir interest in auction research outside of Economics. Though published in an economics journal, the work of Banks, Ledyard and Porter suggests that scheduling and logistical problems (typically the bread and butter of the OR community) could be handled by adapted auction mechanisms. Further, their approach was progressive in the use of decentralized computation, shifting computational burden away from the central decision making entity and to the bidding agents themselves. This "agent-based" decision making has been a hot topic in several fields of research and has strongly influenced the more recent auction literature of Kelly and Steinberg [**32**], who connect this line of research to another government application: universal telephone service obligations. These two works provide a great deal of motivation for our three-stage hybrid auction described in Chapter 4. In particular, we borrow their ideas for demand revelation through user-selected bundle offers, but attempt to improve their methods by diminishing unnecessary computational searching and the ability of bidders to "free-ride" on the revealed demand of others. A more thorough comparison of our hybrid auction to the adaptive user selection technique is provided in Chapter 4.

We note that Ausubel, Cramton, and Milgrom [**4**] also describe a hybrid auction, the Clock-Proxy Auction, developed concurrently with this dissertation. This auction consists of a demand revelation (clock) stage and a sealed-bid (proxy) stage. In the clock stage bidders report their demand for items at the given prices, and prices are adjusted upward until there is no excess demand. At the conclusion of this stage, bidders submit bids for bundles of items to a computer proxy, and an Ausubel-Milgrom [**5**] proxy auction follows, determining an allocation of packages and the corresponding bidder payments.

Potential governmental applications for auctions abound, though the one discussed most prominently in the literature seems to be auctions for the allocation of personal communication spectrum licenses. The sellers of these licenses (such as the Federal Communication Commission (FCC) in the United States) have no significant costs to recover for each license granted, so that the value for such items are determined almost exclusively through competition on the demand-side. More importantly, because several licenses in the same frequency range must be acquired to form a functioning communications network, the value that a particular telecommunications firm places on a given set of licenses is very heavily influenced by which other licenses it has received. This property suggests that a series of single-item auctions would not lead to an efficient outcome in this market and that a more complex auction mechanism should be considered. In the FCC auction-design debate of the early 1990's, it was determined that a combinatorial or package bid auction should not be considered since the general combinatorial auction winner determination problem was computationally intractable (i.e., $\mathcal{NP}$-hard) (see [**45**]). The FCC instead adopted the Simultaneous Ascending Auction (SAA) auction, which allows for simultaneous price discovery without package bidding, foregoing the potential benefits of package bidding due to its computational difficulty. The FCC reported $32 billion in revenue using this approach in 32 auctions between 1994 and 2001 (see [**37**]).

This underlying all-or-nothing assumption that a hard computational problem should either be handled completely or not all is antithetical to the OR approach and has been challenged in this context. As the seminal work of Rothkopf, Pekeč, and Harstad [**62**] points out, many special cases of an $\mathcal{NP}$-hard problem like the combinatorial auction winner determination problem can be solved efficiently, suggesting several applications in which a combinatorial auction can be quickly implemented, including special cases in which the size or type of bundles which may receive bids are restricted without objection from the bidders. The debate surrounding the FCC's design problem helped introduce auction theory to the OR community as a set of complex decision problems, open to heuristic techniques and algorithms tailored to specific classes of instances. The flurry of research that followed paved the way for the current landscape of auction implementation, in which B2B auctions take place using market-specific structure and IP techniques, as discussed below. The FCC seems to have accepted the possible benefits of such a paradigm shift, that a "partial" or "restricted" combinatorial auction may be successfully implemented using limited package bidding and IP solvers, as evidenced by the proposed (and serially postponed) Spectrum Auction #31, discussed, for example, by Günlük, Ladányi, and de Vries [**28**].

Many auctions are complicated by the complex way in which the items combine to form valuable bundles, including auctions for spectrum licenses. A set of spectrum licenses may have a combined value more than the sum of its parts if the licenses geographically cover a particular large region such as the Eastern Seaboard or if they cover a continuous frequency range in a particular region. Furthermore, it may be difficult or unreasonable for the auctioneer to determine in advance which bundles should be considered as valuable since this may vary from bidder to bidder. It is precisely in this situation, where the items at auction have multiple attributes or may be combined differently by different bidders, that the decision problem becomes difficult and advanced combinatorial auction mechanisms become advantageous.

Though spectrum auctions and similarly complex markets dominate the decision-science-type auction literature, auctions are commonly employed for several other governmental allocation problems which receive less attention for their computational difficulty. The most important examples in this group include treasury-bill and electricity auctions, where the auctioned items have fewer distinguishing characteristics and demand can be expressed simply as quantities demanded of a few simple types. Treasury-bills in particular are "commodity-like" and do not combine in complicated ways to form valuable bundles, making advanced auction/decision techniques unnecessary in terms of implementation, though the game-theoretic analysis of these markets may not be trivial (see the work of Ausubel and Cramton [**3**]). It is, however, important that we include treasury auctions among the success stories of government allocation auctions; these markets provide an excellent model for auctioning several "commodity-like" items, especially in situations where diminishing marginal returns dominate.

With spectrum auctions as an example of a market in which items combine to form bundles in a complicated way, potentially benefitting from the use of a combinatorial auction mechanism, and treasury auctions as an example where no such complications arise, making combinatorial machinery unnecessary, deregulated wholesale electricity markets seem to provide a middle ground. On one hand, the items at auction (kilowatt-hours of electrical power supply) are very commodity like, i.e., divisible (simplifying analysis and implementation) and deriving the majority of their value intrinsically, rather than through synergy with other auction items. On the other hand, some synergies do arise in the form of power generator start-up and no-load costs, so that a "pure-commodity" approach may distort a bidding power supplier's ability to communicate true costs of production. This leaves a bidder to either misrepresent her preferences on the side of caution, or to risk dissatisfaction with the results of the auction. The electricity market may therefore be treated with a "tâtonnement" procedure, familiar from the economics literature, modified to handle the non-convexity imposed by the start-up and no-load costs. Market Design, Inc. (www.market-design.com), an auctions consulting firm, has successfully implemented such specialized "clock auctions" for the wholesale electricity markets in France and Alberta, Canada, and have consulted in regional U. S. markets. (Cramton [**14**] provides details.) These regional U.S. markets, such as the Pennsylvania-New-Jersey-Maryland (PJM) system, take a markedly OR approach to electricity allocation, solving Mixed Integer Programs (MIPs) for daily electricity dispatch. The possibility of a centralized restructuring of the U. S. energy market to allocate and price electricity using sophisticated IP-type auction techniques is currently under investigation by the Federal Energy Regulatory Commission. O'Neill et al. [**51**] describe this research.

2.1.2. Auctions and B2B Commerce

While the authorities on the government applications that motivated much of the early combinatorial auction research were reluctant to implement the change to a "package bidding" auction, pioneers of B2B e-commerce quickly recognized the potential benefits of advanced auction techniques and began conducting "combinatorial" auctions as a market intermediary in the mid-1990s. Today, several firms offer auction consulting services, helping a firm conduct an auction for the procurement of transportation lanes, raw materials and other services. Procurement auctions (with a single buyer and several suppliers or sellers, also called reverse auctions) with at least some combinatorial machinery have been conducted by CombineNet [**40**], IBM [**31**], Logistics.com [**43**], Net Exchange [**39**], and Emptoris [**67**]. These B2B auctions assure savings to the procurer through increased price competition and the ability to constrain product quality, delivery-time, and other non-price attributes of the items provided. Conversely, strategic, logistical, and scaling problems facing suppliers are often alleviated with bundle bidding, which may allow expression of volume-discounts, incompatible products/services, and complementary products/services.

In a typical procurement auction, the buyer initiates the auction by inviting quotes from several competing suppliers for a particular set of services (or items). In a combinatorial procurement auction the buyer often has the ability to place other restrictions on the final allocation, such as a lower bound on the number of sellers to ensure diversity and lack of dependence on a particular seller. Other services may be unnecessary unless complementary services are also performed and a buyer may express a need to have a certain subset of services all provided by the same bidding supplier where necessary. On the other end, a supplier may express interest in supplying services only when complementary services are also performed, or may conversely express a desire to only provide a particular service if substitute services are *not* performed. With this wide range of expressability over both substitutes and complements desirable for both buyers and sellers in the proposed set of B2B applications, the optimization approach to winner determination seems to dominate tâtonnement models modified to accept combinatorial information in this context. Indeed, much of the available literature on procurement auctions utilize an IP approach, which is generally flexible enough to accept a wide range of logical statements of preference.

This need for a fully flexible “bid language” is evident in the market for transportation or shipping lanes, in which a buyer requests offers to ship various loads across origin/destination pairs (shipping lanes). In traditional shipping lane markets bidding shippers face uncertainty in the final bundles they receive, making it difficult to forecast their own global shipping schedule, or to express how internal constraints may effect their contractual decision making (e.g. “I can move shipment A or shipment B but not both. Which one should I bid on?). Though the problem becomes too large for IP techniques (and for bidders) when prices are submitted for every possible bundle, it is well known that optimization problems like the winner-determination problem are often more easily solved with the addition of constraints, in this case corresponding to a priori restrictions of the types of bids which may be submitted. Unfortunately, it is difficult to make a prior restrictions on the types of bundles that may be bid on in a shipping lane auction, partly because some of the factors determining the value of a bundle of shipping lanes for a particular shipper may be exogenous to the auction at hand. For example, certain shipping lanes may be complementary to pre-existing contractual obligations of the shipper; a shipper would be especially eager to fill trucks that would otherwise be empty return-trips under existing contracts. Because so many exogenous factors exist and vary from shipper to shipper, a very general bid language seems necessary for this scenario. IP approaches that add restrictions (cuts) only as bid information is received from bidders seem particularly advantageous here.

Sears Logistics held the first large scale auction for shipping lanes using "combined value" or package bids in the mid 1990s with the software support of Net Exchange [**39**]. Their encouraging results suggest that satisfaction with the market mechanism and efficiency may be increased on both sides of the market. Sears Logistics (the procurer of shipping lanes in this case) reported savings of $25 million (13%) in their first combined value auction, while shippers were able to eliminate uncertainty and exposure to winning incompatible or incomplete sets of lanes. With this success documented, it is not surprising that several other large corporations decided to introduce the combinatorial auction paradigm for shipping; Logistics.com reports the implementation of procurement auctions on behalf of Walmart Stores, Compaq Computer Co., Staples Inc., The Limited Inc., and Kmart Corporation [**43**]. In addition, Elmaghraby and Keskinocak [**22**] provide a case study of a successful shipping lane auction for Home Depot.

The acceptance of a combinatorial auction format using IP decision-making techniques for transportation procurement should not be surprising. Internal supply-chain and logistics decisions are often approached from the IP perspective, making the use of IP software for community or market decisions a natural transition. Though less apparent, procurement auctions are finding their way into more general settings, though more reports of success stories may be necessary before they are accepted on a large scale. Some notable success stories include those of CombineNet who report 15% surplus gained by participants in procurement auctions for raw materials (such as coal and steel) and shipping lanes since 2001 [**40**]. In addition, IBM has implemented several combinatorial procurement auctions for Mars, Inc., emphasizing that benefits accrue on both sides of the market (implying overall efficiency gains) and that payback on Mars' investment was less than a year [**31**]. In general, experts advocate that a successful procurement auction may be conducted in markets with a relatively small static group of suppliers, and that mechanism design should emphasize efficiency rather than procurer profits, so as to establish favorable long term relationships. Parkes [**55**] supports this position.

2.2. Auction Theory

Several auctions have been studied and implemented for many years, the English auction and its variants being the most familiar. In an English auction, an item is to be sold (one at a time), but an accurate market price is unknown. In order to determine a price that is satisfactory to both buyer and seller, an auctioneer names successively higher prices, and bidders respond with their willingness to accept these prices, or by dropping out of the auction. The auction concludes when only one bidder is willing to pay the current price. This last bidder receives the item at the price that is a single increment higher than the amount that the next highest bidder is willing to pay. This simple auction is both *incentive compatible* (truth-telling is the best strategy) and *efficient* (the item goes to the bidder who values it the most).

Other long-standing auction formats for a single item include the Dutch auction, in which the price descends until the first bidder bids and wins the item, and the sealed-bid auction, in which bidders each submit a price for the item in question, with the highest bid winning. Auction theorists show that under mild assumptions, the same strategic behavior should be expected for the Dutch and "first-price" sealed-bid auctions, in which the highest bidder pays the amount of her bid. The "second-price" sealed-bid auction (a sealed-bid auction together with the condition that the highest bidder wins the item at the price specified by the second-highest bidder) displays incentive-compatibility. Under an assumption that a bidder's valuation does not depend on the valuation information revealed by her opponents, the second-price sealed-bid auction produces the same outcome as the English auction (see [**34**]).

Figure 2.1. Market Settings For Auctions

Number of Participants	Types of Items	Number of Items	Differentiation
one Seller, many Buyers one Buyer, many Sellers many Buyers and Sellers	Divisible Discrete	Single-Item Multiple-Items	Identical Items Distinct Items

These observations on the equivalence of various auction outcomes, together with the strategic analysis showing that a bidder has no ability to benefit from an untruthful strategy in certain simple auctions, provide the "classical" basis of auction theory. There are two major avenues of contemporary auction theory building on these classical foundations. With a behavioral approach, many pursue more accurate models of bidders to explain the differences between empirical data and theoretical predictions. Others auction theorists design new auction formats for simultaneously auctioning several items, like those proposed in the current work. The challenge of this pursuit is to combine computational feasibility, economic efficiency, and strategic lucidity in a multi-item auction format. Before introducing these research avenues in greater depth, we must first make clear some underlying scenarios and terminology.

2.2.1. Market settings for auctions

Figure 2.1 illustrates the full array of market types for which auctions have been studied. Almost any combination of selections from each column yields a different type of market (the only exception being that the question of differentiation among items has no meaning when only a single item is to be auctioned.) Together this enumerates 18 different market settings to which auctions may potentially be applied. Auctions with a single seller are often referred to as *forward* auctions, as this is the most familiar format. An auction with single buyer and several sellers is usually called a *reverse* auction, but is also known in the literature as a *procurement* auction.

Markets with many buyers and sellers have been called both *exchanges* and *double* auctions. Since there is a large amount of literature concerning exchanges (due to their importance in finance), it may be useful to make a distinction between exchanges and double auctions. Based on current business practices, we propose that an exchange should refer to an open market in which transactions have the opportunity to occur at any time and require only the agreement of the transactors, as in the stock market. A double auction, on the other hand, should refer to markets in which all transactions are decided by a central information collecting entity, adhering to our earlier definition of an auction.

In this dissertation. we maintain a forward auction setting as is common in the literature. As we are interested in areas on the cutting edge of computational difficulty, we focus on auctions for multiple, discrete items. In general we will assume that each item is distinct or distinguishable from any other item.

2.2.2. Single-Item Auctions

Single-item auctions have been studied in economics as games of incomplete information for over forty years [**68**], and may be considered well-understood. Specific topics of interest include several models for the auction of a discrete item with varying assumptions on the behavior of bidders. These models vary according to the amount of information available to each bidder, whether each bidder knows his own valuation for certain or bases this value somewhat on what others think, and the amount of affiliation or correlation among the values of bidders.

For example, the seeming equivalence in outcome between the English and second-price auctions breaks down when bidders' utility is governed by affiliated signals (see [**46**]). In this circumstance the English auction has higher expected revenue, because bidders in the English auction learn over the course of the auction and can bid more competitively with better information. In the second-price auction, on the other hand, lack of information about the value of the item induces conservative bidding. The complete result for single-item auctions is that in terms of maximizing auctioneer revenue, the English auction outperforms the second-price auction which in turn outperforms the first-price auction. Again, Krishna [**34**] provides a good overview of this fundamental material.

As noted above, similar results hold that the Dutch auction may be considered strategically equivalent to the first-price sealed-bid auction. This equivalence is challenged, however, in new work by Carare and Rothkopf [**11**] who show that *slow* Dutch auctions (in which the price drops slowly over the course of a few days in an internet auction) may produce more revenue than the first-price auction (see also [**44**]). As in several of recent papers which challenge long-standing principles of auction theory, we see that the simplifying assumptions used in theoretical work may lead to conclusions which no longer hold when the model is extended to describe the real world more accurately. In the case of Carare and Rothkopf, we see that the value of time must be included as a transaction cost in models of Dutch auctions, and that failure to capture all costs in an auction model may skew a seller's decision making criteria when selecting a particular auction.

Other interesting new topics in the world of single-item auctions include explaining the difference in equilibrium behavior between theoretical work and empirical study. Indeed, real-life bidders do not behave as game theory suggests they should, often not understanding the incentive implications of VCG pricing or the need for proper conditional probabilities to mitigate the "winner's curse". Deltas and Engelbrecht-Wiggans [**18**], and Ding et al. [**20**] provide a glimpse into the field of behavioral economics and its intersection with auction theory. The former provides justification for the persistence of strategically naive behavior, while the latter examines the effect of emotional factors (excitement and frustration) on bidder behavior and auction participation. In both cases the auction format is held to the simple case of a single item in order to study bidders' behavior, which may be very complex despite the simple setting.

2.2.3. Multiple-Item Auctions

The study of multiple-item auctions (or multi-unit auctions) has received greater attention than single-item auctions in recent years, and consequently represents a greater portion of the auction literature reviewed here. Situations proposed for these auctions include forward, reverse, and double auction settings for identical or distinct items that are either discrete or divisible (i.e., all possibilities). Theorists usually assume that results for forward auctions hold for reverse settings and use whichever terminology is convenient for their proposed area of application. Even though many results do apply generally, whether specific auction designs work better for different numbers and types of market participants is territory that has been explored only on an ad hoc basis.

Combinatorial auctions are multiple-item auctions for which bids may be placed on packages of items, and are often referred to as auctions with *package-bidding*. Neither term should be used to refer to multiple-item auctions in general, since many important types of multiple-item auctions need not accept such bids. The Simultaneous Ascending Auction long used by the FCC provides an example of a format that is multi-unit but not combinatorial.

Combinatorial auctions may provide more beneficial outcomes when the value of an item received by a bidder is determined heavily by the other items received. Many applications have been proposed including electricity markets, equities trading, bandwidth auctions, transportation exchanges, pollution right auctions, auctions for airport landing slots, supply chains, and auctions for carrier-of-last-resort responsibilities for universal services [**65**]. As mentioned above, innovative vendors already host a significant amount of B2B commerce using combinatorial auctions, making the study and development of more specialized combinatorial auctions a lucrative pursuit, despite the computational difficulties to bidders and auctioneers alike.

Certain developments in the study of multi-unit auctions in general can be attributed to economic reasoning similar to the results surrounding single-item auctions. The behavior of models which extend these ideas to the case of distinct items is less well understood. Much of the analysis attending multi-item multi-round auctions rely heavily on assumptions of substitutability among the items being auctioned. Complementarity of items is a widespread phenomenon in markets for which multi-unit auctions are proposed, and the existence of superadditive prices (among complements) is a strong reason for expecting that multi-unit auctions might produce higher revenues than separate single-unit auctions. These strong substitutability assumptions should therefore be treated with some suspicion.

Figure 2.2. Auction design components

Component	Typical Choices	Objectives
Winner Determination	Provisional Winners with Stopping Rules Solve an IP formulation	Maximization of Revenue & Ex-post Satisfaction
Payment Determination	Pay-As-Bid Uniform Vickrey-Clarke-Groves	Incentive Compatibility Revenue Maximization
Information Flow	Sealed-Bid Price Offers Dynamic Price w/Demand Reporting	Privacy Preservation Cost of Elicitation
Bid Language	All-or-Nothing Package Bids XOR-of-OR	Minimize Exposure Problem & Threshold Problem

It can be shown that the "Walrasian" approach may not converge without these very special utility functions displaying the "gross substitutes property" [**66**]. The reason for this difficulty becomes clear when we consider that the problem of finding a revenue maximizing allocation of items in the general combinatorial auction is $\mathcal{NP}$-hard. The substitutability conditions on multi-unit auctions in the economics literature have been proven to be equivalent to a submodular restriction on each bidder's indirect utility function [**5**], assuring that various iterative algorithms will quickly converge to global solutions. This suggests that the difficulty of the general combinatorial auction problem is one of optimization over a non-convex feasible region, with complementary packages causing local peaks in utility.

The $\mathcal{NP}$-hardness of the winner determination problem opens the door for an algorithmic approach to combinatorial auctions. As with any $\mathcal{NP}$-hard problem, few hope of finding a general procedure which can be guaranteed to work for all situations. Instead, computer science and OR researchers focus on identifying situations for which winner determination is easy (as in [**62**]), specializing solution techniques to various market structures (as in [**28**]), or developing approximation algorithms for the general case (as in [**16**]). The computationally-minded literature often seeks to take advantage of special structure and special techniques to design a multi-unit auction tailored to a specific market. To better understand the developments that are being made, we introduce the categories of auction design components, tabulated in Figure 2.2. Though every auction design must define rules for each design component, in many auctions the choices may not be unusual. For example, determining the winner and payment in a first-price, sealed-bid, single-item auction is a trivial matter, and the information flow and bid language are as simple as possible. Still, it is precisely the variations on these four auction attributes that fuel the growing literature on the design of multi-unit distinct-item auctions, including the contributions of this dissertation.

2.2.4. Winner Determination

Variations on the winner determination problem are the most common innovations in the algorithmic approach, using different IP formulations and different algorithms for solving these various formulations. This is both because the computational complexity lies in winner determination, and because most innovations in the other components dictate at least some adjustment of the winner determination process. Günlük, Ladányi, and de Vries [**28**], and Sandholm et al. [**65**] demonstrate that some advances in the field may simply be new computational methods tailored to solving the winner determination problem of a specific auction. Xia, Stallaert, and Whinston [**69**] show, however, that general IP techniques (CPLEX) may dominate these specialized search methods. De Vries, Schummer, and Vohra [**17**] provide a deep exploration of the connections between auction winner determination and the general theory of optimization, recognizing several auction winner determination techniques as either primal-dual or subgradient optimization.

Winner determination is impeded not only by the $\mathcal{NP}$-hardness of the general problem, but also by exponential growth of input relative to the number of items N. We will refer to this as the *exponential bundles problem*: there are $2^N - 1$ nontrivial packages in an auction of N items, too many for a human to consider explicitly for an auction of more than a few items. As in deVries and Vohra [**16**], our review of the literature leads us to distinguish two general methods of restricting the size of input in order to alleviate the exponential bundles problem of winner determination, *restricted-subset methods* and *restricted-preference methods*.

Since GWD allows too many subsets to be specified for even a modestly large number of items, some methods suggested in the literature employ an auction in which bids are only accepted on a certain set of permissible packages of items, necessarily much smaller than the set of all of subsets of I. Rothkopf, Pekeč, and Harstad [**62**] provide several such *restricted-subset* combinatorial auctions for which polynomial-time algorithms exist. Though these methods work well for very specific market structures, there are two problems limiting their applicability.

First, the auctioneer decides in advance which packages may be bid on; this may be unrestrictive if the items being bid on have very specific uses and synergies may only be derived in very particular ways. But when the bidders are firms competing for raw materials or governmental licenses which can be utilized in different combinations according to varying technologies and market strategies, a mechanism in which the bidders themselves choose the packages may be more desirable.

Secondly, if a bidder does not submit a bid on a particular package, how much should she for that package if it is awarded to her? If the auction dictates that this package may not be awarded to her, the mechanism may sacrifice efficiency; it considers a package of non-zero value to have zero value and may thus miss an optimal allocation. If instead the mechanism relaxes the single-bundle constraints it negates expression of substitutability among packages. This is the case in [**62**]; a bidder may receive a bundle together with other bundles at an additive cost. A third suggested resolution may be to set $b_j(S) = \max_{S' \subseteq S} b_j(S')$, but this only effectively increases bids on items that auctioneer would otherwise be giving away. It seems fair to assume that an auctioneer is not inclined to give items away for free, and sets reserve prices that are effectively positive bids on each singleton set (i.e., each set containing only one item).

Remark: There is a general trade-off inherent in any model for which a price is not specified for each bundle: either a bidder may not receive a bundle for which he has not specified a price, sacrificing efficiency; or he may receive the union of subsets he has bid on at the sum of the corresponding prices, sacrificing expression of substitutability.

An alternative to the restricted-subset methods is given by the *restricted-preference* methods. The full generality of expression afforded in unrestricted winner determination is surely more than is necessary for almost all practical applications. A restricted-preference method places limitations on what kind of bidding functions may be used, based on assumption or inference on the behavior and preferences of bidders. Rather than reducing the number of bundles that can be bid on in any arbitrary fashion, these methods place limitations on the relationship among the values assigned to various bundles.

For example, under most reasonable circumstances it is safe to assume a *non-decreasing* preference restriction: $b_j(S \cup \{i\}) \geq b_j(S)$. If bidder j is given one more item i she is not strictly worse off, and may be forced to bid accordingly. This particular objective function restriction is not strong enough to alleviate much of the computational burden, and the few restrictions explored in the literature that are strong enough to positively effect computations typically do not allow for the expression of complementary bundles [**16**].

In this dissertation we explore two new restricted-preference methods and their effects on winner determination, particularly on our ability to avoid the exponential bundles problem. In Chapter 3, we see that bid tables are quite restrictive in terms of expressability, but greatly simplify winner determination. In Chapter 6, we see that matrix bids provide a less restrictive format, increasing compactness of preference expression, but not greatly diminishing the computational difficulty from the general winner determination problem.

2.2.5. Payment Determination

Unlike winner determination, methods for payment determination vary little over the current frontier of multi-item auction design. This may be because of the widespread acceptance of the class of Vickrey-Clarke-Groves (VCG) mechanisms for honesty-inducing payment determination (for primary sources see [**13**],[**25**],[**68**]). Chen et al. [**12**], for example, do question the subtleties of applying this paradigm, and show that some IP formulations of winner determination can lead to inflated payments when using a VCG mechanism naively. Indeed, despite its theoretical beauty, several authors expose drastic problems with VCG payment mechanisms, explaining their scarcity of implementation (see [**5**],[**61**],[**63**], and [**64**]). Among these problems for VCG mechanisms are the vulnerability to false-name bidding, collusive interference, bid-taker cheating, and failure of the payments to support a core outcome, further questioning the widespread acceptance of the VCG prices.

Despite these apparent drawbacks, VCG mechanisms maintain several interesting properties as a means of payment determination and greatly simplify the analysis of computational methods. Noteworthy among these properties is the connection between linear programming dual prices and the calculation of VCG payments (see [**8**] or [**55**]).

Another central question is how to select a core outcome when VCG payments do not belong to the *core*. In general, the core refers to outcomes in a centralized decision-making scenario for which no coalition of participants would prefer to break away from the central mechanism to achieve a different outcome among themselves. (The core is discussed in depth for the case of one-sided auctions in Chapter 5.) iBundle [**53**] and the ascending proxy auction [**5**], for example, both give an alternative to VCG pricing with methods that achieve bidder-Pareto-optimal payments within the core under certain convexity conditions. Parkes, Kalagnanam, and Eso [**56**] explore how prices may be achieved that approximate VCG payments as closely as possible, preserving some portion of the incentive compatibility while maintaining budget-balance constraints in a combinatorial double auction. The analogous procedure for one-sided auctions is a bit trickier, however; approximating VCG payments subject to core constraints is more difficult than subject to budget-balance, because the former requires an exponential number of constraints. In Chapter 5 we discuss this in greater depth and explore a new method for finding bidder-Pareto-optimal payments in the core via constraint generation, without the need for restrictive convexity assumptions. In addition, we discuss the qualitative selection of a core pricing scheme from the set of Pareto-efficient core outcomes where no Pareto dominance can be established.

2.2.6. Information Flow

In addition to their potential role in payment determination, supporting dual prices may also play a part in the information flow structure of some iterative (i.e., multi-round) multi-unit auctions, serving as feedback to inform bidders how to proceed in the next round (see [**28**]). Similar feedback prices are determined each round by solving optimization problems in the new formats of both Kwasnica et al. [**35**] and Kwon and Anandalingam [**38**]. Both of these works represent innovative new designs in the information flow of auctions, though they have conflicting ideas of how updated price information should be captured and presented (i.e., the nature of the information flow).

In Chapter 3, we see that a primary strength of the Bid Table approach is that linear prices are easily computed for each round of a Dynamic Bid Table Auction and fed back to the bidders to inform their bid strategies for the upcoming rounds. We show that the computed linear prices support a Walrasian equilibrium and are thus informative of the competitive value of the auctioned items in the absence of positive synergy.

As a general note regarding information flow, there is a central dichotomy among the different approaches to information flow, dividing the set of all auctions into two types. In one type bidders are asked to submit demand functions (or demand correspondences, more generally), either in response to some current price vector or for a larger set of prices. Single-item English auctions are the simplest auctions of this type, where each demand function reported at the current price is simply willingness to buy or not. The second general information flow type contains auctions in which the submitted information consists of price offers for various bundles. Though the sealed-bid auctions do fall into this category, use of this type of information transmission is also common in dynamic multi-unit auction design.

Though the relationship between these two types of submission are readily seen to be inverse (bundles assigned to prices as opposed to prices assigned to bundles), the contexts can be quite different and may not exhibit the same dynamic behavior. A similar interpretation in the language of optimization is that the two approaches are dual. If in a primal model's decision variables tell whether a given item is chosen, then certain dual variables may be interpreted as prices for items.

2.2.7. Bid Languages

This duality of auction formats draws the first major distinction in the classification of *bid languages*. In single-item auctions it seems to be the only distinction: do bidders submit demand at a given price (the language of indirect mechanisms) or a reservation price above which demand is zero (the language of direct mechanisms)? Models of multi-unit auctions usually assume a straightforward generalization of one of these two languages, but there are reasons for studying alternative bid languages. Because of the difficulty of expression imposed by the exponential bundles problem, various indirect and direct mechanisms for preference revelation are studied in the multi-item auction literature. Many of these approaches build bid expressions through the use of *flat bids*.

Definition: A *flat bid* $(S, p) \in I \times \mathbb{R}^{\geq 0}$ is a bid of price p on itemset S, with no bid on any other bundles.

The direct language considered in the most general combinatorial auction settings (including the multi-unit VCG auction) involves bidders submitting a reservation price for every single bundle. The auctioneer is not allowed to combine bids at the sum of the values in this context; the acceptance of any one bid excludes the acceptance of all others from that particular bidder. This language leaves no room for an exposure problem, but the exponential bundles will be experienced in full force for bidders using this XOR (of flat bids) language, even with only a modest number of items. We are interested therefore in a language by which bidders may express their preferences compactly in large multi-unit auctions.

One approach to this problem is a language which receives flat bids and relaxes the constraints in GWD reflecting that only a single bundle may be awarded to a bidder. In this setting a bidder may receive a bundle she did not bid on, made up of smaller bundles which she did bid on, at the sum of the corresponding prices. Though this OR language (of flat bids) does receive some attention in the literature, this approach on its own does not allow bidders to express any amount of subadditive preferences (e.g. two items are substitutes) and may result in demand reduction where substitution effects are significant. This elucidates one of the strongest reasons for exploring the bid languages of this dissertation: price-vector agents are designed specifically to counter the exposure problem among substitute items, a problem that is ignored by other bid languages.

In general, the problem of bid language design for combinatorial auctions is to mitigate the exponential bundles problem by devising a system for bid submission in which a single bid "sentence" may simultaneously place bids on multiple bundles. Many approaches use flat bids joined by logical connectives, usually OR and XOR. In any logical language, the basic building blocks which cannot be decomposed into a smaller meaningful expressions of preferences are called *atomic bids* or *bid atoms*. For example, in a language using flat bids as bid atoms, one could bid (\$300 on $\{A, B\}$) XOR (\$400 on $\{C, D\}$), using the exclusive XOR to keep from getting both bundles.

Nisan [**50**] describes the strengths and weaknesses of such languages in detail. He develops OR^*, a language using ORs of flat bids and the possible addition of "phantom items" to the auction by a bidder to cause an exclusion between two or more bids that would otherwise not share any items. Though arbitrary sentences in OR and XOR cannot be converted polynomially among one another, a sentence from any of the languages he considers (including OR-of-XOR and XOR-of-OR of flat bids) can be neatly expressed in the OR^* of flat bids language.

A weakness of the languages studied by Nisan [**50**] is their inability to express the "k-of preference", i.e., "I will pay \$$p$ for any k items from the set S." Describing this bid in any of the languages studied by Nisan requires a the use of an exponentially long sentence. Boutillier and Hoos [**10**] extend the languages of Nisan to include k-of statements and logical statements on *items* as atomic bids. These authors explore various algorithms for solving the winner determination problem in the context of the resulting languages using stochastic local search in [**30**], but unfortunately expand the k-of statements into flat bids in order to run their algorithm. Only in the later work of Boutillier [**9**] is an IP approach for winner determination explored that handles k-of statements in the same "concise" form in which they are easily written (without expansion into an exponentially larger format prior to optimization.)

In Chapter 6, we propose a bid language using matrix bids as bid atoms. We show that any of the bid atoms described for other languages can be expressed within a single matrix bid, and that often several atomic bids from the existing languages can be packed into a single matrix bid. Given that the k-of operator can be used compactly within the matrix bid language, only Boutillier [**9**] can claim to have a language that is as expressive as matrix bidding while maintaining polynomial size in the number of items. With the added ability to express hard capacity constraints, capacity costs, price differentiation and conditional add-on values within the same atomic bid, we argue that matrix bidding may offer a superior alternative to languages such as that of Boutillier [**9**], requiring fewer bid atoms because more can be said in a single atom. Further, the associated winner determination problem can be solved as an assignment problem with side-constraints, allowing for rapid solution of large-scale instances.

2.2.8. Equilibrium Behavior in Multiple Item Auctions

Much of the multiple-item auction literature focuses on the computational issues of winner/payment determination, bid submission language, and information updates, as outlined above. This is due to the influx of computer science and OR scholars since the interesting computational problems became widely known in the 1990s. Still, the roots of auction theory are well entrenched in the economic field of game theory. Auctions are games of incomplete information, and theoretical studies explain what competitive behavior may be expected in equilibrium.

As noted above, there is not yet a general consensus on how to tackle the various computational difficulties of multiple-item auctions, and it seems that perhaps several multi-unit formats will survive academic scrutiny and be selected for actual implementation. It should come as no surprise that game theory has had only a limited ability to accurately describe the behavior of the multitude of formats, since each auction presents a unique game for analysis. It does of course seem reasonable from a research standpoint to wait and see which formats are still standing when the dust settles, so that time is not wasted studying approaches which may turn out to be inferior.

One auction that has received special attention is the simultaneous ascending auction (SAA), due in part to its use in FCC spectrum auctions. Engelbrecht-Wiggans and Kahn [**23**] investigate the equilibrium properties of this auction, providing a model which explains the observed existence of low-revenue outcomes. Their results concern the need for eligibility rules and careful reserve pricing in order to reduce bidders' ability and incentive to collude. They describe equilibrium behavior in which the small number of bidders are able to anticipate the auction outcome and decide not to compete. In this way they are able to "divide up the pie" without driving up the prices on one another. A similar (and more general) model of this behavior is given by Ausubel and Cramton [**3**].

Another set of detailed data illustrating collusive bidding in the SAA is given by Cramton and Schwartz [**15**]. Their analysis focuses on signaling among competitors as a form of tactical collusion to achieve lower payments at the termination of the auction. While competing in simultaneous auctions, bidders are seen to play punishments strategies: bidder 1 submits a high bid on item A (which she is not interested in) in order to drive up the price for bidder 2, the likely winner of A, as punishment for submitting a competitive bid on item B, which bidder 1 will likely win. To be sure that bidder 2 gets the message, bidder 1 abuses the precision of expression afforded to the bidders, encoding her initials or an indication of the market for item B in the smaller digits of the punishment bid. A bidder employing this strategy may find it advantageous to “name-tag” all her bids with a code in the smallest digits, so that other bidders can identify her and “stay off her turf.”

Though a convincing argument is made for the use of anti-collusive measures in multiple-item auctions in general, these principles have scarcely been applied to the combinatorial auctions setting. It seems reasonable to expect that much of the intuition from the simultaneous ascending auction does carry over into the realm of package bidding: using information revealed over the course of the auction, bidders may anticipate the final allocation and curtail competition to achieve the outcome at low prices.

The three-stage design outlined in Chapter 4 is designed to make it difficult for this type of equilibrium to be achieved. We will show how an initial Bid Table auction may proceed with only linear price information returned to the bidders. Since these price signals we compute for each item are not set at the bid amount of the winner, it is impossible to send signals in the added significant digits as described above. Similarly, we will show how to conduct a second stage allowing bidders to express demand on complementary packages without exposure, and with no strict need for the winners of various bundles to be announced publicly. We feel that this greatly diminishes a bidder’s ability to forecast the auction outcome and “tactically” or “passively” collude to achieve a low-revenue equilibrium. The auction terminates in a third sealed-bid stage, which is designed to find the efficient solution without making as much information available for deceptive strategies in earlier stages. In addition, we describe how “honesty constraints” can be made on bidding in Stage III to encourage demand revelation for bundles in Stage II.

CHAPTER 3

Bid Table Auctions

Demand revelation is desirable in many auction environments, and in multi-item auction environments bidders would often like to see *linear prices* for each item as information about the current state of the auction. Such prices allow a bidder to determine the price for a bundle by simply adding the values for the items in that bundle, making it quite simple for bidders to evaluate a number of potential packages. In order for these prices to be an accurate indicator of demand in the auction, we would like to have prices that are *accurate* and *separate* winning bundles from losing bundles. If the auction were to close immediately, we want the current prices to reflect the actual payments made by winners (accuracy) and for the current prices to be higher than the losers are willing to pay (separation). Unfortunately, linear prices which are accurate and separate are not always possible in a combinatorial auction when items are complements, as demonstrated by the following example.

Example: In a four-bidder, three-item auction let the bids on items A, B, and C be as follows:

$$b_1(\{A,B,C\}) = 6, \qquad b_2(\{A,B\}) = 5$$

$$b_3(\{A,C\}) = 5, \qquad b_4(\{B,C\}) = 5$$

Clearly, the efficient solution is to award all three items to bidder 1, but what prices can be assigned to the individual items that separate the winner from the losers? In order for the losing bidders to be satisfied, the sum of the prices of the items in a bundle should exceed any losing bid on the bundle:

$$p_A + p_B \geq 5$$

$$p_A + p_C \geq 5$$

$$p_B + p_C \geq 5$$

But this implies $p_A + p_B + p_C \geq 7.5$, a total payment that is too high for bidder 1, who will pay at most 6.

This example illustrates the well known failure of linear prices. At the conclusion of a combinatorial auction allowing for the most general expression of preferences, prices can only be expressed in terms of bundle-payments made by the winners and cannot be decomposed into meaningful individual item prices. Though this is true at the termination of a combinatorial auction, we argue that linear prices may still play a role in the early stages of a combinatorial auction, where they may be used as signals of the value of items if taken individually, before positive synergy is taken into consideration.

In this chapter, we present a new auction format based on the familiar assignment problem and show that this is a natural environment for the use of linear item prices. On one hand, if the bidders do not find the preference restrictions of the format to be restrictive, we find that this is a compact format which can be used to find either VCG outcomes (discussed in §3.2) or minimal Walrasian equilibrium individual item prices (discussed in §3.3). The latter can be implemented in either a one shot submission of bid tables or through iteratively updated bid tables in a Dynamic Bid Table Auction, guided at each stage by linear item prices.

If, on the other hand, we are in environment requiring a more robust expression of preferences (e.g. complements are present), we argue that bid tables provide a restricted format for which linear prices make sense. In Chapter 4, we introduce a multi-stage combinatorial auction using a Dynamic Bid Table Auction as an initial demand revelation phase. Bid tables allow us to utilize the attractive linear prices as long as it makes sense to do so, until any further bidding requires a combinatorial element to mitigate positive-synergy exposure. Relative to the SAA, this can be accomplished without negative-synergy exposure, and without the ability for competitors to signal among themselves.

We begin this chapter with a discussion of the basic properties of bid table expression.

3.1. Assignment Preferences and the Gross Substitutes Property

Our multi-item auction will have N unique items for sale, with the set of items denoted $I = \{1, 2, ...i, ...N\}$, and the set of M bidders will be referred to as $J = \{1, 2, ...j, ...M\}$. For any bundle or package of items $S \subseteq I$, each bidder j perceives some value $v_j(S) \geq 0$ equal to her utility of receiving this bundle at zero cost. Further, in this treatment we assume non-negativity ($v_j(S \cup \{i\}) - v_j(S) \geq 0$, $\forall i, S$) and define a bidder's net utility as $u_j(S, p) = v_j(S) - \sum_{i \in S} p_i$, where each component p_i of price vector p denotes a price for item i (we assume quasilinear net utility).

Next, we say a bundle of items S is *demanded by j at a price p* if S maximizes net utility for bidder j at p. By denoting the set of all bundles demanded by j at price p as $D_j(p)$, we may write this mathematically as:

$$D_j(p) = \arg\max_{S \subseteq I} u_j(S, p)$$

A bidder may demand several bundles at p if each maximizes net utility, and demands $\emptyset$ when prices are too high. Additionally, we assume $v_j(\emptyset) = 0$, $\forall j$.

In a *Bid Table Auction*, each bidder j has a set of A_j agents he potentially wishes to satisfy, denoted by the set $K_j = \{1, 2, ...k, ...A_j\}$. We assume that each item in the auction may be consumed by only one agent and that each agent can consume at most one item. To motivate the proposed use of the Bid Table Auction as the first stage of a landing slot application, it is useful to think of each agent as representing a potential flight, and each item as a slot made available to an airline in an airport for landing a single plane. Often in Chapter 4, the items in the auction will be referred to as slots, but for the remainder of Chapter 3 we maintain a general attitude with respect to auction items, believing that there are likely other applications for Bid Table Auctions.

Let v_{ijk} denote the utility perceived by bidder j when agent k receives item i. The preference assumption of the Bid Table Auction can be specified as follows:

$$\text{(AP)} \qquad v_j(S) = \max \sum_{i \in S} \sum_{k \in K_j} v_{ijk} \cdot x_{ijk}$$

$$\text{subject to} \quad \sum_{i \in S} x_{ijk} \leq 1, \forall k \in K_j$$

$$\sum_{k \in K_j} x_{ijk} \leq 1, \forall i \in S$$

$$\text{where } x_{ijk} = \begin{cases} 1 & \text{if item } i \text{ is assigned to bidder } j\text{'s } k\text{th agent} \\ 0 & \text{otherwise} \end{cases}$$

In short, a bidder's value for a bundle of items is a maximal value assignment of those items to her agents. Bidders modeled in this fashion may be said to have *assignment preferences,* as their preferences are governed by an assignment network. As is well known, the constraint matrix of this integer program is total unimodular, indicating that the problem is solved to integral optimality by its LP relaxation. This implies that a bidder could rapidly determine her value for any set of items with an LP solver or faster combinatorial algorithm designed specifically to solve assignment problems, even for a large value of N.

A bidder with assignment preferences can express his value for every bundle of items in a *bid table* containing the values of v_{ijk} with a row for each item i and

Figure 3.1. Assignment Preferences in a Bid Table

	agent 1	*agent* 2	*agent* 3
Item a	0	0	2
Item b	2	5	3
Item c	4	6	4
Item d	0	3	0

column for each agent k. For example, with the bid table of Figure 3.1 we may determine a bidder's value for any bundle of items a, b, c, or d if assigned to any of three possible agents. Considering the values in this bid table, we notice that a bidder's value for any single item is simply the maximum value for that row in the bid table; if a bidder is awarded only one item, the agent that experiences the most utility from this item will be accommodated. Thus item a by itself is worth 2 units to the bidder, item b by itself is worth 5 units, item c by itself is worth 6 units, and item d by itself is worth 3 units. The value for some collection of items is not, however, necessarily equal to the sum of his values for individual items, as there may be conflicts when the row maximums occur in the same column. For example, for bidder j with this bid table $v_j(\{b\}) = 5$, $v_j(\{c\}) = 6$ but $v_j(\{b, c\}) = 9 \neq 5 + 6$. We find $v_j(\{a, b, c, d\}) = 11$ with an optimal assignment of a to the third agent, b to the second agent and c to the first agent. We notice that to achieve this amount we do not assign item c to the second agent, despite this being the overall highest value in the bid table, and also that we choose to assign item a and not item d, despite the fact that $v_j(\{a\}) < v_j(\{d\})$.

These observations show that assignment preference valuation functions can have some non-intuitive properties and are not contained in the class of additive valuation functions (in which the value of some set of items always equals the sum of the values for the individual items). We notice that any additive valuation function can be modelled by a bid table by taking constant rows and at least as many agents as there are items. Thus additive valuation functions are properly contained in the class of assignment preference valuation functions. One may next ask how assignment preferences relate to the larger class of valuation functions which satisfy the gross substitutes property.

Definition: The *gross substitutes property* holds if and only if the following condition holds for every bidder j: For any price vectors $p' \geq p$ with $p' \neq p$, and any $S \in D_j(p)$, there exists a set $S' \in D_j(p')$ with $i \in S'$ for all $i \in S$ with $p'_i = p_i$.

Or roughly, if the prices rise on some of the items in a demanded set, then there is at least one demanded bundle at the new prices still containing all previously demanded items for which the price did not increase.

The gross substitutes property is sufficient to guarantee the convergence of several ascending price multi-item auction formats to a Walrasian equilibrium (e.g. those of Ausubel and Milgrom [**5**] or Gul and Stacchetti [**27**]) and has other beneficial properties which will be discussed below. Though it is a common assumption due to its attractive theoretical properties, it is uncommon for theorists to describe applications which give rise to this property, to explain under what conditions it is a safe assumption to make, or to suggest how to enforce this restriction among bidders if it is necessary for the convergence of an auction to a desirable outcome. In this dissertation, we start instead from the assumption of assignment preferences and show that the gross substitutes property follows as a result.

The following proof of gross substitutes under assignment preferences is constructive. Using an LP solver to determine demanded bundles under assignment preferences may give any demanded bundle after a price increase and will often not provide the bundle that verifies the validity of the theorem. For any demanded bundle S^* before a price increase and a given demanded bundle S' after the price increase, the algorithm described in the proof constructs a new demanded bundle S^{**} after the increase, containing all previously demanded items in S^* for which the price has not risen.

Theorem 3.1. A valuation function determined by assignment preferences displays the gross substitutes property.

Proof. Suppose we begin at price vector p^1 with a particular demanded bundle $S^* \in D_j(p^1)$ and consider price vector $p^2 \geq p^1$, with $p^2 \neq p^1$ and a particular demanded bundle $S' \in D_j(p^2)$. Under assignment preferences a demanded set is always supported by an assignment of items to agents which maximizes total value net of prices (sometimes more than one supporting assignment may exist). Select particular assignments (sets of arcs) A^* and A' supporting sets S^* and S', respectively. We will say that arc $i \to k$ is in A if assignment A has item i assigned to agent k.

Now, given an element $\bar{\imath} \in S^*$ with $p^1_{\bar{\imath}} = p^2_{\bar{\imath}}$ that is not in S', we show how to find a new set $S^{**} \in D_j(p^2)$ with $\bar{\imath} \in S^{**}$ and corresponding assignment A^{**}. We do this by showing how to construct two sets IN and OUT such that $A^{**} = (A' \backslash OUT) \cup IN$.

Algorithm: Constructing a set $S^{**} \in D_j(p^2)$ containing $\bar{\imath}$
Step 0: Set $InItem = \bar{\imath}$, $OutItem = IN = OUT = \emptyset$.
Step 1: Let k be the agent assigned $InItem$ under A^*.
Step 2: Find the item (or $\emptyset$) $OutItem$ with $OutItem \to k$ in A'.
Step 3: Add arc $OutItem \to k$ (or $\emptyset$) to OUT. Add arc $InItem \to k$ to IN.
Step 4: If $OutItem \notin S^*$ (or $OutItem = \emptyset$), terminate, else set $InItem = OutItem$ and goto Step 1.
Terminate: Set $A^{**} = (A' \backslash OUT) \cup IN$.

Claim: The set S^{**} implied by arc set A^{**} is in $D_j(p^2)$. Suppose not. Then $S' \in D_j(p^2)$ provides $u_j(S^{**}, p^2) < u_j(S', p^2)$. Since by construction each member of OUT was in A', this yields $u_j(IN, p^2) < u_j(OUT, p^2)$, with the obvious interpretation of the utility for an arc set. That is

$$u_j(IN, p^2) = \sum_{i \to k \in IN} v_{ijk} - \sum_{i | i \to k \in IN} p_i^2 < u_j(OUT, p^2) = \sum_{i \to k \in OUT} v_{ijk} - \sum_{i | i \to k \in OUT} p_i^2$$

But because every item assigned under IN is assigned under OUT, except for $\bar{\imath}$ which does not experience a price change from p^1 to p^2, we have $u_j(IN, p^1) < u_j(OUT, p^1)$. But by construction of these arc sets $(A^* \backslash IN) \cup OUT$ is a feasible assignment, and then we must have $u_j(A^*, p^1) < u_j((A^* \backslash IN) \cup OUT, p^1)$, contradicting our choice of $S^* \in D_j(p^1)$.

The validity of the claim provides a demanded set containing $\bar{\imath}$. Each execution of the algorithm finds a set in $S^{**} \in D_j(p^2)$ containing an element from S^* that was missing in S' and does not remove any elements from $S^* \cap S'$. By repeating this procedure (setting $S' = S^{**}$ each time) we arrive at a set in $D_j(p^2)$ containing all the desired elements of S^*. □

To illustrate the idea of the proof consider the following example in which the items in the auction are denoted by lowercase letters. Suppose at prices p^1 the bidder in question demands the bundle $\{a, b, c, d, e, f, g\}$, which is found to maximize utility with the assignment of items to agents $A^* = \{a \to 1, b \to 2, c \to 3, d \to 4, e \to 5, f \to 6, g \to 7\}$. Suppose next that prices rise on items a and d. The bidder then recomputes for an optimal bundle at these prices and finds the optimal assignment $A' = \{h \to 1, r \to 2, p \to 3, q \to 4, g \to 5, d \to 6, c \to 7\}$, but this assignment clearly does not validate the gross substitutes property; items b, e, and f did not experience a price increase but they do not appear in the newly demanded bundle.

To apply the algorithm described in the proof, first consider finding a demanded set including b. Putting A^* above A' and designating destination agents by column, we see below that agent 2 who was assigned b in A^* is assigned r in A':

$$
\begin{array}{ccccccccc}
A^* & : & a & b & c & d & e & f & g \\
 & & & \downarrow & & & & & \\
A' & : & h & r & p & q & g & d & c \\
- & - & - & - & - & - & - & - & - \\
A^{**} & : & h & b & p & q & g & d & c
\end{array}
$$

The manipulation performed by the algorithm is in its simplest form here. The value of A^{**} at p^2 must be at least as much as the value of A' at p^2, or else it would be possible to switch r in for b in A^* and receive a higher value at p^1, contradicting the optimality of A^* as a demanded bundle. This reasoning only holds because the price of b does not change in the movement from p^1 to p^2, and because r can be switched in freely as it is not in A^*.

When we try to find a demanded bundle at p^2 containing e, it is not quite so easy; the agent assigned e under A^* is assigned g under A', and since g is already assigned under A^* a one-for-one switching argument fails. The algorithm rectifies this by tracing back a path until an item is found that was not allocated under A^*. For example, the following diagram helps us see that in moving from A^* to A' (which has now been replaced by A^{**} from the previous step) item e's agent is reassigned item g, whose agent in A^* is reassigned item c, whose agent in A^* is reassigned by item p which was unallocated in A^*. The same optimality argument can be used to show a demanded bundle at p^2 including item e; $\{p \rightarrow 3, g \rightarrow 5, c \rightarrow 7\}$ can be replaced by $\{c \rightarrow 3, e \rightarrow 5, g \rightarrow 7\}$ or else the optimality of A^* is contradicted.

$$\begin{array}{ccccccccc}
A^* & : & a & b & c & d & e & f & g \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
A' & : & h & b & p & q & g & d & c \\
- & - & - & - & - & - & - & - & - \\
A^{**} & : & h & b & c & q & e & d & g
\end{array}$$

Similar arguments allow us to settle on a final set A^{**}, containing all items which were demanded at price p^1 for which prices did not increase. The final step is displayed

Figure 3.2. Preferences that do not fit in a Bid Table

$$v(\{a\}) = 10, \quad v(\{a,b\}) = 20$$
$$v(\{b\}) = 12, \quad v(\{a,c\}) = 19$$
$$v(\{c\}) = 13, \quad v(\{b,c\}) = 17$$

a	10	8?
b	12	10?
c	13	

in the following diagram.

$$\begin{array}{ccccccccc} A^* & : & a & b & c & d & e & f & g \\ & & & & & \downarrow & & \downarrow & \\ A' & : & h & b & c & q & e & d & g \\ - & - & - & - & - & - & - & - & - \\ A^{**} & : & h & b & c & d & e & f & g \end{array}$$

Theorem 3.1 assures us that the gross substitutes property holds when bidders are restricted to the use of bid tables only, as in Stage I of the auction proposed in Chapter 4, or any isolated Bid Table Auction implementation. To complete the characterization of assignment preferences with respect to the gross substitutes property, we note that the valuation function of Figure 3.2 maintains the gross substitutes property but cannot be expressed as an assignment preference valuation function. The gross substitutes property follows from the submodularity and positivity of the valuation function. To see that these preferences cannot be expressed as an assignment preference valuation function (or equivalently, as a bid table) notice that the value of any two items is less than the sum of the values for the individual items, thus the values for the individual items must all occur in the same column of a bid table. But now if we attempt to put in a value of 8 into column 2, row a, or a value of 10 into column 2 row b to express $v(\{a, b\}) = 20$, we either overvalue the bundle $\{a, c\}$ at 21 or the bundle $\{b, c\}$ at 23. There is therefore no way to express these gross substitute preferences as assignment preferences.

With Theorem 3.1 and earlier observations we have the following corollary, where V_{add}, V_{AP}, and V_{sub} denote the classes of valuation functions that have additive preferences, assignment preferences, and gross substitute preferences, respectively.

Corollary 3.2. $V_{add} \subset V_{AP} \subset V_{sub}$

Though all containments in Corollary 3.2 are proper, we note that assignment preferences do retain some of the interesting properties lost between V_{sub} and V_{add}. Notably, we can embed in a Bid Table Auction an example from Gul and Stacchetti [**27**] that demonstrates the VCG outcome may have lower payments than the lowest Walrasian Equilibrium. This embedding is shown in Figure 3.3, providing an example of a Bid Table Auction for which the VCG payments are strictly less than any Walrasian Equilibrium. In one efficient allocation, bidder X gets $\{A, D\}$, bidder Y gets $\{B\}$ and bidder Z gets $\{C\}$, with VCG payments of 12, 2, and 2, respectively. The lowest

Figure 3.3. A Bid Table Auction for which VCG payments are lower than in any Walrasian equilibrium

Bidder X

A	8	0	0	0
B	8	0	0	0
C	0	8	0	0
D	0	8	0	0

Bidder Y

A	6	0	0	0
B	0	6	0	0
C	2	0	0	0
D	0	0	0	0

Bidder Z

A	0	0	0	0
B	2	0	0	0
C	0	6	0	0
D	6	0	0	0

Walrasian equilibrium price vector is however, $p_A = p_B = p_C = p_D = 6$, charging more to bidders Y and Z than in any VCG outcome. Having established that the VCG outcomes and the lowest Walrasian Equilibrium may differ in a Bid Table Auction, we now discuss Bid Table Auctions which achieve each of these outcomes, in §3.2 and §3.3 respectively.

3.2. VCG Bid Table Auction Implementations

Corollary 3.2 allows us to apply the following theorem:

Theorem 3.3. (Ausubel and Milgrom [**5**]) For an auction with a single unit of each item and bidders with valuation functions drawn from the set V such that $V_{add} \subseteq V$, the following conditions are equivalent:

(1) $V \subseteq V_{sub}$

(2) For every profile of bidder valuations drawn from V, adding bidders can never reduce the seller's total revenues in the VCG auction.

(3) For every profile of bidder valuations drawn from V, any shill bidding is unprofitable in the VCG auction.

(4) For every profile of bidder valuations drawn from V, any joint deviation by losing bidders is unprofitable in the VCG auction.

In the sealed-bid Vickrey-Clarke-Groves (VCG) auction ([**13**], [**25**], [**68**]) each bidder submits her true valuation for every possible bundle of items. A winning allocation is then determined which distributes bundles to bidders so as to maximize total value. To assure that each bidder has the incentive to reveal her total value honestly, she pays not her reported value for the bundle she receives, but this value less an appropriate discount. This VCG discount assures that a bidder does not pay any more than would be necessary to receive this bundle given her opponents honest reports, and is equal to the value of the final allocation minus the maximum value attainable without this bidder. A bidder only decreases her chances of receiving her efficient allocation of items by misreporting, with no possible gain.

The assurance of honest valuation reporting is well known to be the strength of the VCG auction, but as Ausubel and Milgrom [**5**] point out, there are several drawbacks. Shill bidding occurs when a bidder enters a false identity into the auction in a way that results in an inefficient allocation or alternative set of payments which is preferred by the deceitful bidder. Similarly, Ausubel and Milgrom demonstrate how two or more losing bidders in a VCG auction may increase their bids to become winning without having to pay for this increase. These problems with the VCG auction are explored in further depth elsewhere [**63**], but we note that these difficulties rely on the existence of complementary items, an impossibility when the gross substitutes condition holds. Theorem 3.3 ensures that these problems of collusive behavior and revenue reduction sometimes recognized as drawbacks to the VCG auction have no effect when the gross substitutes property holds.

Since the hypotheses for Theorem 3.3 and condition (1) of Theorem 3.3 are satisfied according to Corollary 3.2, we may conclude that the Bid Table Auction scenario is one for which the VCG auction attains its full strength. The advantages of the Bid Table Auction scenario over the general VCG auction context are that each bidder may simply submit a bid table to express her preferences (a method that is far more compact than issuing a price for every one of the $2^N - 1$ possible nonempty bundles) and the winner determination problem can be solved polynomially using LP techniques. Both of these features are important for an auction of many items. Notice that at LaGuardia airport, for example, over 800 slots may be available in a single day; an auction for these slots which enumerates all possible bundles would be quite impossible.

In order to run a *VCG Bid Table Auction*, one must compute both an efficient allocation and the set of VCG payments for all bidders. To determine an efficient allocation, the auctioneer, having received bid tables containing the submitted bids of b_{ijk}, needs to solve the LP:

$$\text{(P)} \qquad z_J = \max \sum_{(i,j,k)\in I\times J\times K_j} b_{ijk} \cdot x_{ijk}$$

$$\text{subject to} \quad \sum_{i\in I} x_{ijk} \leq 1, \; \forall (j,k) \text{ with } j \in J \text{ and } k \in K_j$$

$$\sum_{j\in J} \sum_{k\in K_j} x_{ijk} \leq 1, \; \forall i \in I$$

$$x_{ijk} \geq 0, \; \forall i, j, k$$

To determine VCG payments, the auctioneer can solve this problem again without bidder j to find the appropriate discount for bidder j, $z_J - z_{J\setminus j}$, where $z_{J\setminus j}$ denotes the value of the objective value of P with bidder j removed. The assignment corresponding to z_J determines a bundle S_j with bid value $b_j(S_j)$ for each bidder. Each bidder in the VCG auction receives S_j and pays $b_j(S_j) - (z_J - z_{J\setminus j})$. For the entire auction at most $M + 1$ LPs must be solved. Since there are $O(MN^2)$ variables and $O(MN)$ constraints in each LP, and because it is well known that LPs can be solved in polynomial time (as a function of the number of constraints and variables), we conclude that the VCG Bid Table Auction can be solved in polynomial time. There are in fact combinatorial algorithms for solving assignment problems (of which this is an instance); see Ahuja, Magnanti and Orlin [**1**] for a $O(M^2N^3 + MN \log MN)$ time algorithm.

Though Theorem 3.1 allows the application of Theorem 3.3, implying that because of the gross substitutes property the Bid Table Auction is a good preference format for a VCG implementation, it also limits the types of possible applications of the Bid Table Auction to those in which there is no complementarity among items. In Chapter 4 we discuss an auction for airport landing slots, where bidders (airlines) perceive the constraints of the Bid Table Auction (inability to use a slot for more than one flight and inability to land a flight without a slot), but may also perceive some complementarity among slots (e.g. being able to offer several flights to the same destination in one day may offer extra profit potential, as discussed above). As described in Chapter 4, we develop an iterative multi-phase auction, beginning with a *Dynamic Bid Table Auction* used to reveal preferences for slots as substitutes, with complementarity expressed only later in the auction.

3.3. Dynamic Bid Table Auctions

Ignoring for now the issue of complements (i.e. maintaining the assumption that bidders have assignment preferences), it seems that the primary drawbacks to the VCG Bid Table Auction are the lack of price discovery and privacy preservation. Though privacy preservation may in principle be legally enforced, lack of price discovery may be an inherent concern and discourages the use of a sealed-bid VCG Bid Table Auction. A bidder may not know how to fill out a bid table with honest valuations for agent/item pairs without knowing her opponents' values for various items, and would prefer a dynamic auction to learn about her competition.

Towards the goal of implementing a dynamic auction for the landing slot application, we examine the LP dual to P:

$$\text{(D)} \qquad \min \sum_{i \in I} p_i + \sum_{j \in J} \sum_{k \in K_j} s_{jk}$$

$$\text{(3.1)} \qquad \text{subject to} \quad p_i + s_{jk} \geq b_{ijk}, \; \forall i, j, k$$

$$p_i \geq 0, \; \forall i \in I$$

$$\text{(3.2)} \qquad s_{jk} \geq 0, \; \forall j \in J, \; k \in K_j$$

Problem P has integer optimal solutions, and an optimal solution to problem D will have the same objective value. This dual formulation suggests a set of "supporting prices" p_i for each item. If we stipulate that a bidder receiving item i in an optimal solution of P pays p_i, the value of s_{jk} becomes the surplus perceived by bidder j's agent k (abbreviated (j, k)).

The complementary slackness conditions for the primal dual pair P-D are:

$$\text{(3.3)} \qquad \begin{aligned} \forall i, j, k \quad x_{ijk} > 0 &\Rightarrow p_i + s_{jk} = b_{ijk} \\ p_i + s_{jk} > b_{ijk} &\Rightarrow x_{ijk} = 0 \end{aligned}$$

$$\text{(3.4)} \qquad \begin{aligned} \forall j,k \qquad \textstyle\sum_{i\in I} x_{ijk} < 1 \Rightarrow s_{jk} = 0 \\ s_{jk} > 0 \Rightarrow \exists i \text{ such that } x_{ijk} = 1 \end{aligned}$$

$$\text{(3.5)} \qquad \begin{aligned} \forall i \in I \quad \textstyle\sum_{j\in J} \sum_{k\in K_j} x_{ijk} < 1 \Rightarrow p_i = 0 \\ p_i > 0 \Rightarrow \textstyle\sum_{j\in J} \sum_{k\in K_j} x_{ijk} = 1 \end{aligned}$$

Each of these conditions (presented in equivalent pairs) carries an economic interpretation reinforcing the validity of the model. If agent (j, k) is awarded item i, condition (3.3) implies $p_i + s_{jk} = b_{ijk}$, validating our reference to s_{jk} as surplus. If an agent is not awarded an item in an optimal solution, by condition (3.4) we have $s_{jk} = 0$, and then the constraints (3.1) become $p_i \geq b_{ijk}$, $\forall i \in I$; if an agent is empty-handed at optimality, the optimal dual prices make any item too expensive for this agent to buy. Similarly, the second statement of (3.4) says that if agent (j, k) perceives any surplus, then it must be the case that (j, k) received an item. Further, if we evaluate the potential surplus that item i would bring to agent (j, k) who receives a different item at optimality, we find $s_{jk} \geq b_{ijk} - p_i$; the price of i is great enough that a change from the item awarded at optimality to i for agent (j, k) does not increase surplus. The last pair of conditions, (3.5), state simply that an item will have a nonzero price only if it is received by some agent.

Definition: An allocation of bundles to bidders $(S_1, S_2, ...S_j...S_M)$ and price vector p constitute a *Walrasian equilibrium* if and only if for every bidder j, $S_j \in D_j(p)$.

Theorem 3.4. Assuming truthful demand reporting and assignment preferences, the allocation $(S_1, S_2, ...S_j...S_M)$ given by an optimal solution to P together with prices for items $p = (p_1, p_2, ...p_i...p_N)$ given by the corresponding dual solution to D constitutes a Walrasian equilibrium.

Proof. Suppose not: there is some bidder j and some bundle $\overline{S_j}$ with j strictly preferring $\overline{S_j}$ to S_j. Equivalently,

$$b_j(\overline{S_j}) - \sum_{i\in\overline{S_j}} p_i > b_j(S_j) - \sum_{i\in S_j} p_i$$

where $b_j(S_j)$ and $b_j(\overline{S_j})$ are supported by agent sets K and $\overline{K}$, respectively. Complementary slackness conditions (3.3) provide $b_{ijk} - p_i = s_{jk}$ for each item i assigned to j in S_j, yielding $b_j(S_j) - \sum_{i\in S_j} p_i = \sum_{k\in K} s_{jk}$. Similarly, constraints (3.1) yield $\sum_{k\in\overline{K}} s_{jk} \geq b_j(\overline{S_j}) - \sum_{i\in\overline{S_j}} p_i$. Together this implies

$$\sum_{k\in\overline{K}} s_{jk} > \sum_{k\in K} s_{jk}$$

but with $s_{jk} = 0$ in the optimal solution of D for any agent $k \notin K$, and $s_{jk} \geq 0$ for all tasks, we have

$$\sum_{k\in\overline{K}\cap K} s_{jk} > \sum_{k\in K} s_{jk}$$

a contradiction. □

Among all dual solutions there exists one solution that is bidder-optimal, the lowest Walrasian equilibrium. The existence of a Walrasian equilibrium under the gross substitutes property is established by Kelso and Crawford [**33**]. The uniqueness of this bidder-optimal Walrasian equilibrium when the gross substitutes property holds follows naturally from the work of Gul and Stacchetti [**26**], who demonstrate the lattice structure of Walrasian Equilibria under gross substitutes. In our case, this existence and uniqueness is guaranteed because the gross substitutes property holds by Theorem 3.1. It is known from the work of Demange, Gale, and Sotomayor [**19**] that versions of the "Hungarian algorithm" (a primal/dual method for solving assignment problems) yield this lowest Walrasian equilibrium in the special case that $A_j = 1, \forall j$.

As in Demange, Gale, and Sotomayor [**19**], the Hungarian algorithm finds an efficient allocation in a Bid Table Auction and the prices used in its solution provide an optimal solution to the dual problem D, together forming a Walrasian equilibrium, from Theorem 3.4. In general, however, this method produces Walrasian prices that are greater than or equal to the actual lowest Walrasian equilibrium prices for a Bid Table Auction. This is because according to formulation D alone, the price a bidder pays may be determined by one of her own agents, as if own agents must price-compete among themselves.

To avoid this self-competition problem when finding the lowest Walrasian equilibrium, we introduce the following Dual Pricing Problem DPP. This formulation specifies an LP-characterization of the minimal Walrasian equilibrium prices for a Bid Table Auction. Specifically, given a solution to the primal problem P above with objective value z, we may fix the dual objective from D at its optimal value in constraints (3.6) and maximize total surplus over all bidders.

$$\text{(DPP)} \qquad \max \sum_{j \in J} \sum_{k \in K_j} s_{jk}$$

$$\text{(3.6)} \qquad \text{subject to} \quad z = \sum_{i \in I} p_i + \sum_{j \in J} \sum_{k \in K_j} s_{jk}$$

$$\text{(3.7)} \qquad p_i + s_{jk} = b_{ijk}, \ \forall i, j, k \text{ with } i \rightarrow j, k$$

$$\text{(3.8)} \qquad s_{jk} = 0, \ \forall j, k \text{ with } \emptyset \rightarrow j, k$$

$$\text{(3.9)} \qquad p_i + s_{jk} \geq b_{ijk}, \ \forall i, j, k \text{ with } i \nrightarrow j$$

where we expand our use of the $\rightarrow$-notation to express assignment under a specifically chosen efficient solution to problem P; for example, $i \rightarrow j, k$ expresses that item i is awarded to bidder j's kth agent in the selected efficient allocation. Similarly, $i \nrightarrow j$ signifies that item i is not awarded to bidder j.

We note that the most important distinction between DPP and D comes in constraint set (3.9) in which only constraints that do not involve self-competition are enforced. That is, form the set of constraints (3.9) from constraint set (3.1) of formulation D by removing all constraints $p_i + s_{jk} \geq b_{ijk}$ involving both an item i and its winner, bidder j. This approach is equivalent to re-solving the primal allocation problem P and the dual problem D with a Hungarian primal/dual method after lowering all non-winning entries in a winning row of a bid table to zero. We use the formulation DPP in the following proof as it provides an interesting interpretation as a price adjustment procedure; starting with an efficient allocation and the optimal solution of D, lower the prices on winning bidders as long as no one complains, any violated constraint from (3.9) interpreted as a possible complaint from another bidder. Knowing that prices may be easily and transparently adjusted to a unique minimal Walrasian equilibrium is a primary benefit of Theorem 3.1 (gross substitutes). We will use this unique price vector in a multi-round setting, using Bid Table Auctions as a mechanism of demand revelation with distinct meaningful price signals at each round of submission. We now prove that we achieve these desirable price signals formally.

Theorem 3.5. Given an optimal solution to the primal problem P with objective z, solving DPP to maximize bidder surplus yields the lowest Walrasian equilibrium price vector p^*.

Proof. Begin with price vector p^1 which is an optimal solution to D and therefore a Walrasian equilibrium price by Theorem 3.4. Note that this optimal solution to

D gives a feasible solution to DPP with constraints (3.7) and (3.8) holding by the strong duality of P and D. Lowering any component p_i^1 and raising the corresponding component s_{jk}^1 by the same amount (where i is assigned to j, k in the solution to P) will have no effect on constraints (3.6), (3.7) or (3.8).

Claim: If such a shift from price to surplus does not violate constraints (3.9) the resulting price vector continues to support a Walrasian equilibrium. If not, some other bidder $\bar{j}$'s agent $\bar{j}, k$ not assigned item i would prefer item i to whatever item that has been assigned to $\bar{j}, k$ (if any). This implies that $b_{i\bar{j}k} - p_i > s_{jk}$ which would violate the corresponding constraint from (3.9).

We therefore proceed to shift price to surplus until any possible shift causes a violation of some constraint from (3.9), achieving a price vector p^2.

Claim: p^2 is the lowest Walrasian equilibrium price vector. Suppose not: let $p^3 \neq p^2$ be the lowest Walrasian price vector. From the lattice theory of Walrasian equilibria when the gross substitutes condition holds (as established by Gul and Stacchetti [**26**]) this Walrasian Equilibrium price is unique, and for every component $p_i^3 \leq p_i^2$. Since $p^3 \neq p^2$ there must be some i for which $p_i^3 < p_i^2$. Since our price-to-surplus shifting procedure has terminated, a shift from p_i^2 to p_i^3 must violate some constraint from (3.9), thus we have the following inequality holding for some j, k with i not assigned to bidder j:

$$p_i^3 + s_{jk} < b_{ijk}$$

But now at price vector p^3, bidder j who is allocated bundle $S_j \not\supseteq \{i\}$ prefers bundle $S_j \cup \{i\}$ to S_j, implying that p^3 does not support a Walrasian equilibrium, a contradiction. □

Now that we have shown how the lowest Walrasian price vector will be computed at each round, we propose the *Dynamic Bid Table Auction* proceeding in multiple rounds as follows. Accept bid tables from all bidders and determine a winning allocation (solution to P) using any technique (e.g. the Hungarian algorithm). Adjust the prices as suggested in Theorem 3.5 to find an optimal solution to DPP, and therefore the set of unique lowest Walrasian equilibrium prices. Announce winning prices and allow bidders to adjust their bid tables subject to the following rules:

- Any bid in a non-winning row of a bid table must be raised at least to the current price for that item (row) plus one price increment, or else it may not be altered for the remainder of the auction.
- A bidder who does not wish to increase an entry to the required amount may increase it to a price below the current price plus one increment. This is the bidder's "last-and-best" offer.

The auction then continues by computing a new set of winning bids and prices, and the process repeats until no one wishes to raise any bid table entries any further.

To show that this procedure progresses to a desirable equilibrium, assume that each bidder perceives a set of maximum bid table values, v_{ijk}. We would expect to see these entries from honest bidders in the direct revelation VCG as in §3.2, and now show that the dynamic game converges to the same outcome as the direct revelation game, given some assumption of straightforward bidding. In this Dynamic Bid Table Auction scenario, we will say that a bidder bids straightforwardly if she increases a bid table b_{ijk} in a non-winning row by the minimal increment whenever the potential surplus $v_{ijk} - p_i$ for agent (j, k) is greater than the actual current surplus for agent (j, k), $\bar{s}_{jk} = v_{\bar{\imath}jk} - p_{\bar{\imath}}$, where $\bar{\imath}$ is the item currently awarded to agent (j, k). Here the modifier *actual* and 'bar' over the s emphasize that this is the actual surplus as perceived by the bidder, not the apparent surplus which may be computed using her revealed information: $s_{jk} = b_{\bar{\imath}jk} - p_{\bar{\imath}}$. In either case, this surplus will be zero if no item is awarded to agent k. We also assume that if a straightforward bidder is asked to raise a bid table value above v_{ijk}, then she will raise it to v_{ijk}, reporting her true valuation as she loses eligibility to alter the entry further.

Theorem 3.6. A Dynamic Bid Table Auction with straightforward bidders terminates at an efficient equilibrium of the direct revelation Bid Table Auction, with the unique lowest Walrasian equilibrium prices.

Proof. Part A: First, we show that the allocation at termination is an efficient outcome of the direct revelation game. Suppose not. Let A be the allocation at termination of the dynamic auction with current bid table values b_{ijk}. By supposition there is an allocation $\bar{A}$ such that

$$\sum_{\bar{A}} x_{ijk} \cdot v_{ijk} > \sum_{A} x_{ijk} \cdot v_{ijk} \tag{3.10}$$

where the summation over an allocation signifies summation over all i, j, k with values of x_{ijk} given by that allocation. Because we have assumed that bidders bid straightforwardly and that the dynamic auction has terminated, for any bid table column j, k which is allocated item i under allocation A and $\bar{\imath}$ under $\bar{A}$, it must be the case that $v_{\bar{\imath}jk} - p_{\bar{\imath}} \leq v_{ijk} - p_i$ (if not the straightforward bidder would want to continue bidding on $\bar{\imath}$). This inequality also holds (reflexively) for any agent j, k that is awarded the same item under both allocation A and $\bar{A}$. For any j, k allocated item $\bar{\imath}$ under $\bar{A}$ which is not allocated an item under A, this condition becomes $v_{\bar{\imath}jk} - p_{\bar{\imath}} \leq 0$. Finally, by individual rationality we also have that $0 \leq v_{ijk} - p_i$ for any i allocated to j, k under A (particularly we take this inequality for any columns j, k which are for agents receiving items in both allocations, allocated items under A but not under $\bar{A}$). We then sum these three sets of inequalities, selecting (and multiplying by x_{ijk}) the appropriate one for each agent j, k who receives items in either allocation A or $\bar{A}$, or both, yielding:

$$\sum_{\bar{A}} x_{ijk} \cdot (v_{ijk} - p_i) \leq \sum_{A} x_{ijk} \cdot (v_{ijk} - p_i)$$

$$\sum_{\bar{A}} x_{ijk} \cdot v_{ijk} \leq \sum_{A} x_{ijk} \cdot v_{ijk} \tag{3.11}$$

with the second inequality following since each item is allocated exactly once in each allocation, allowing us to cancel the sum of all p_is from each side. But (3.11) contradicts (3.10), with the desired result following.

Part B: Next we show that the prices at termination of the dynamic schedule auction are the lowest Walrasian equilibrium prices for the direct revelation schedule auction. Using Theorem 3.5, this is equivalent to showing that the solution to the pricing problem DPP using the value $\bar{z}$ (computed using values from v_{ijk}) has equal values for all p_i to the solution of DPP using value z (computed using b_{ijk}).

Given that the optimal allocation is unchanged by increasing all b_{ijk} values to their reservation point v_{ijk} from Part A, increase all b_{ijk} values to v_{ijk} and simultaneously increase every surplus s_{jk} by the same amount wherever i is allocated to j, k. We claim that this provides a solution to DPP using value $\bar{z}$ with identical values of all p_i, as desired. Clearly, the simultaneous shift of s_{jk} values with b_{ijk} values upholds DPP constraints (3.6) and (3.7). Constraints (3.8) are upheld since s_{jk} only changes for columns which win items, while a violation of a constraint from (3.9) would imply a violation of a termination condition. Since none of the constraints of DPP are violated, our new solution must be feasible to DPP. Since any increase of the DPP objective function must be accompanied by an equivalent increase to the primal objective z, and because we have achieved all such increase as surplus, we may be assured that our new objective function is optimal. Since no p_i values have changed we have the same lowest Walrasian equilibrium price vector at the termination of the Dynamic Bid Table Auction and in the direct revelation Bid Table Auction, as desired. □

Part B of this proof demonstrates the desirable privacy-preservation property of the Dynamic Bid Table Auction relative to the static revelation version: the actual surplus each bidder perceives need not be revealed in the dynamic case, only that surplus is at least one increment. By making the bid increment smaller we may in this way approach maximal privacy-preservation.

The results of this section hold under the assumption that each bidder's preferences are modeled accurately in bid tables, an assumption which we have shown to be more restrictive than the gross substitutes property. Recent trends in auction design suggest that the use of auctions which work well under the gross substitutes property (e.g. those of Gul and Stacchetti [**27**] or Ausubel [**2**]) may be used to reveal information necessary for bidders to make decisions in an auction allowing for more general expression of preferences, those for which gross substitutes does not hold. At the end of a Dynamic Bid Table Auction, each bidder should be comfortable that he has bid enough on individual items (time-slots in the airport scenario) without being exposed to the risk of paying too much for substitute items. When this Dynamic Bid Table Auction is used as the first stage in the multi-stage auction described in Chapter 4, the bidders should not have exposed themselves to the risk of paying too much for an item or set of items in an attempt to secure a package of complements, being assured of participation in later stages which allow for expression of positive synergy.

CHAPTER 4

A Three-Stage Auction for Airport Landing Slots

It has been suggested recently that a multi-item auction may more efficiently allocate airport landing slots at certain high-density airports, some of which are already slot-controlled (LaGuardia, Washington National, Kennedy International, and O'Hare International). Currently, administrative practices and "grandfathering" procedures are used to determine which airlines have the right to land at these airports, though these measures have led to apparent inefficiencies, such as the "baby-sitting" of slots with small inefficient flights in order to maintain landing rights under the FAA's "use-it-or-lose-it" policy [**6**]. Since a similar rejection of the inefficiency and potential inequity of an administrative allocation caused the FCC's adoption of the SAA, it seems natural to consider a similar change for the FAA's allocation of controlled landing slots.

In this chapter we lay the groundwork for a new three-stage auction design suited to the landing slot application. We demonstrate several of its features and explore how it may be tailored to fit this application specifically. Though the general version of this auction may be of interest for use in other applications, we first explore the features of the landing slot scenario to motivate the structure of the model in

§4.1. §4.2 gives an overview of the entire three-stage auction process, with §4.3 and §4.4 providing details for Stages II and III, respectively.

4.1. Auctioning Landing Slots

An extensive examination of the problem of allocating scarce landing slot rights is given by the consulting firm DotEcon [**21**] who argue strongly in favor of a market based allocation mechanism over an administrative procedure. Ball, Donohue, and Hoffman [**6**] provide a similar investigation of the situation in the U.S., with a focus on safety considerations and suggestions for auction design. Rather than reiterate these arguments to motivate the potential benefits of a landing slot auction, we refer the interested reader to these sources for motivation of the problem, and only briefly explore the auction design problem based on several features advocated in that literature.

We first note that there is reason to believe that the demand for the rights to take-off or land an aircraft at a given airport may be adequately controlled by the regulation of only one of these movements. Since an aircraft cannot depart from an airport without first arriving, the volume of departures will be regulated implicitly when the right to land is properly regulated. Further, in-air delays are more expensive than ground delays; thus it makes sense that landing rights should be the more clearly defined and tightly controlled of the two movements.

For safety reasons the exact departure and landing times must be left in the hands of air-traffic control, though the volume of possible air movements in a given interval or block of time may be controlled without specifying an exact time. The items for sale in the auction will therefore be rights to land in a particular short interval (say 15 minutes) at a particular slot-controlled airport. Owners of such rights will be held responsible for landing within this period and may be penalized for gross or regular non-adherence to this schedule. In case of poor weather this time period may be altered by means of ground delay under the authority of air-traffic control.

As suggested by Ball, Donohue, and Hoffman [**6**], it will be beneficial to have at least two, and potentially three markets for landing slot rights: a long-term market where airlines purchase rights to land at a particular airport, held well in advance of actualization; a secondary market held when demand and scheduling becomes known, a month or so in advance of actualization, allowing airlines to buy or sell slots from one another as needed; and a third market for day-of-operations exchanges when weather and cancellation information is known. In this chapter we focus on the first market, in which airlines buy long term leases to land during a particular time of day, on a particular day of the week, in a particular season, repeatedly for 5 to 10 years. In this market, airlines establish their long term operational strategy and stake their claim to provide a particular service or control a certain share of the consumer market.

In this chapter we suggest a new combinatorial auction design for this long-term market. Few, if any, available auction designs are capable of handling the very large number of slots that would be up for sale at a single airport. At least eight hundred time slots are utilized in a day at LaGuardia according to the Official Airline Guide (OAG), while the FCC has been skittish about auctioning more than a dozen spectrum licenses in a combinatorial format. The seminal work of Rassenti, Smith and Bulfin [**59**] suggests a network wide auction of links between airports, but their simulations endorsing the use of a combinatorial format make use of only six items at auction. Because there are only a handful of airports currently under slot restrictions, we feel justified in restricting ourselves to an auction at a single airport where the number of items at auction may be several hundred, rather than deal with the possibility of around a hundred thousand in a full network auction. Also, at the recent NEXTOR workshop [**49**] devoted to the use of market mechanisms for congestion management in aviation, there seemed to be a general consensus that market mechanisms should be introduced one congested airport at a time.

As noted above, the desire for combinatorial auctions results from "synergy" in the preference structure of a bidder, which may be positive when the bundle contains mostly complements, or negative if the bundle contains mostly substitutes.

What kind of synergy would we expect to find in a long-term airport landing-slot auction? We argue that there are natural reasons for expecting airlines to find landing slots as both substitutes and complements, adding a level of complexity that must be reflected in the auction design.

Examination of airport scheduling information in the OAG reveals two potential forms of complementarity among bundles of slots which stem from the basic operational strategies of the airlines. The first is that of "banking" in which airlines schedule several planes to land around the same time (usually at a hub location) in order to collect the passengers together and regroup them on connecting flights. This will add value to certain bundles of closely grouped slots (positive synergy in the eyes of bidders with this strategy).

We also recognize a second apparent complementarity among bundles of slots which may be referred to as "shuttling". Airlines gain efficiency and market power by providing a shuttle service between major airports. Airlines with this operational strategy in mind will value bundles with regularly spaced time slots more than the sum of the individual slots, another source of positive synergy.

Negative synergy in the proposed market for landing slots comes from the logistical constraint that a flight may only use one time slot. If an airline is only interested in a single flight at the airport in question, for example, then the value of a bundle of more than one time slot is the value of the most valued individual slot.

This negative synergy due to conflict may persist when a larger number of flights is considered. Suppose airline j has two potential flights to be considered, high profit flight-1 which is valued at 10 regardless of which of slots A or B it lands in, and a lower profit flight-2 which is valued at 7 in either slot A or B. We then have $v_j(\{A\}) = v_j(\{B\}) = 10$ (since either one on its own can accommodate flight-1), but $v_j(\{A, B\}) = 17 < v_j(\{A\}) + v_j(\{B\})$, confirming the negative synergy. This example elucidates a general phenomenon of negative synergy due to the underlying assignment constraints that each airline must face. In order to protect a bidder from negative-synergy exposure, our multi-stage auction begins with a price-vector agent format (i.e. bid tables) and continues to respect that structure for the duration of the auction.

The interplay between the Dynamic Bid Table Auction, itself a price-vector agent implementation, and the other stages of the auction are outlined in the following section that introduces the basic framework of the auction mechanism we propose for airport landing-slot auctions, called a *Schedule Auction*.

4.2. Schedule Auction Framework

The mechanism we propose proceeds in three stages, as shown in Figure 4.1. The first two stages are designed to reveal price information while mitigating the *exposure problem*, a phenomenon recognized as a primary reason for the use of combinatorial auctions. If items are awarded singly an honest bidder revealing price

Figure 4.1. Outline of the Schedule Auction

Stage	Format	Problem Treated
I	A Dynamic Bid Table Auction	negative-synergy exposure
II	Bidders "probe" particular packages for a price, Set a reservation value for unsuccessful probes	positive-synergy exposure
III	Auctioneer conducts proxy auction with "honesty constraints" from Stage II	threshold, free-rider

information on individual items *exposes* the bidder to paying too much for some sets of items. The *negative-synergy exposure problem* occurs when a bidder reveals honest price information about substitute items and is exposed to the risk of paying too much for items that are worth less when taken together (negative synergy). The *positive-synergy exposure problem* occurs when a bidder reveals honest price information about complementary items and is exposed to the risk of paying too much for an incomplete set of complements (unrealized positive synergy). In both cases cautious bidders will have an incentive to not tell the truth and may be reluctant about revealing price information.

The first stage of our auction addresses the negative-synergy exposure problem. By using the Dynamic Bid Table Auction introduced in Chapter 3, a bidder is guaranteed the ability to simultaneously raise bids on alternative slots for the same flight without being forced to purchase more than one slot for any flight. Similarly, a bidder may simultaneously raise bids on slots for some alternative collection of

Figure 4.2. A Bid Table for landing slots

	DTW	MIA	ATL 1	ATL 2
9:00	10	0	0	0
10:00	15	0	0	0
11:00	10	0	0	0
12:00	0	15	0	0
1:00	0	17	0	0
2:00	0	0	18	12
3:00	0	0	18	12
4:00	0	0	18	12
5:00	0	20	0	0

flights and be assured of winning only one. The use of bid tables for airport landing slots is illustrated in Figure 4.2.

Recall that the essential feature of a bid table is that only one bid may be accepted from each row (associated with a single item) and only one from each column (associated with each flight or a collection of flights). So for example this bidding airline may want bring in a morning flight from DTW and may express that it is worth more to have the 10:00 time slot than the 9:00 or the 8:00. This airline may do so as illustrated in the first column without running the risk of purchasing more than one of these slots. Similarly this airline may have two flights which exclude each other, say an early afternoon flight from MIA or a 5:00 flight from MIA. By putting the bids for both of these flights in the second column the airline guarantees that it will not be subjected to purchasing time slots for both flights. The third and fourth columns illustrate a more sophisticated possibility: the airline is interested in running two mid-afternoon flights from ATL, but wants to pay less for the second one. By bidding as indicated in Figure 4.2, the airline will pay at most 18 for the first of the mid-afternoon slots and 12 for the second afternoon slot it receives.

Stage I of the auction proceeds by bidders submitting and adjusting bid tables like the one in Figure 4.2. Following the submission of bid tables by all bidders the auctioneer announces a current winning allocation and a price for each item. These prices will be dual prices from the primal allocation problem which are equal to the lowest Walrasian equilibrium prices (as shown in §3.3). Bidders then adjust and resubmit their bid table entries according to the following eligibility rules:

- an entry may not decrease
- an entry in a non-winning bid table row must be raised to at least the current price for that row plus one increment, or else a "last and best" entry may be made that cannot be adjusted again for the remainder of the auction
- entries in a winning row of a bid table need not be adjusted.

Such rules are necessary for this stage to ensure that the auction proceeds at a healthy pace and so that bidders may not lay in waiting without revealing their intentions.

When a round occurs in which no bidder chooses to increase any bid table entry, the auction enters Stage II, designed to mitigate the positive-synergy exposure problem by accepting all-or-nothing package bids. In this stage each bidder is asked in turn to submit a package of items and a price which guarantees to increase auction efficiency if accepted. If this bid survives the remainder of the auction, the bidder receives the items in the package, paying no more than the specified price, as well as any items she may still be winning individually from Stage I at the prices determined there. Because a particular package may be intended for a specific collection of flights, a bidder is allowed to specify the bid-table columns (agents) associated with these flights so that the auctioneer may remove them from her bid table. This will ensure that multiple slots are not won for the same flight, one from the bid-table and one from the package bid.

Using the techniques described in §4.3, each bidder in Stage II can "probe" the market to determine what price must be offered for a particular bundle of items. Once he has found a subset for which he is willing to pay the necessary amount, he may submit this package/price pair which is guaranteed to be winning, until (possibly) it is "knocked out" by another bidder's package bid. We suggest that probing may proceed in a round-robin fashion, where the order of bidding is generated randomly for each round. Further, if the results of Stage I (i.e. the final bid tables) and each of the provisionally winning Stage II bids are released to the bidders, they may continue to probe while waiting their turn. This will allow bidders to explore more scenarios and potential business plans, so that many package bids may be evaluated in preparation for a turn to bid. If instead bidder privacy is deemed more valuable in a particular application, the provisionally winning Stage II bids and final Stage I bids maybe known only to the auctioneer, but then more time should be given to each bidder for probing, since this would be the bidders' only way to learn about the competition.

An important feature of our model is the order in which different types of preference information is revealed over the course of the auction, postponing the solution of difficult computational problems until they are assured to be necessary. For example, the all-or-nothing package bids of Stage II introduce a high level of computational difficulty (we discuss the $\mathcal{NP}$-completeness of the feasibility problem with such bids in §4.3). By first executing a Dynamic Bid Table Auction in Stage I, we eliminate from consideration any package bid which doesn't at least beat the "individual prices" for the items in the package. We place Stage I before Stage II in order to drive up these prices on individual items as much as possible before considering any package bids, eliminating as many as possible from consideration since they are more difficult to handle. We then shift the computational burden to the bidder who must decide which bundle to bid on in Stage II.

This idea of first driving up individual prices and then shifting computational burden to the bidders is present in the PAUSE auction of Kelly and Steinberg [**32**]. However, the use of SAA in their own Stage 1 does not address the negative-synergy exposure problem, unlike the Bid Table Auction. Indeed, we expect that if a Dynamic

Bid Table Auction were simply used to replace the SAA in Stage 1 of Kelly and Steinberg's model, bidders would experience less negative-synergy exposure and be able to bid more aggressively on individual items. The resulting higher prices on individual items will in turn ease more of the computational burden of package bidding.

We note that the recent hybrid auction design of Ausubel, Cramton, and Milgrom [**4**] provides similar advantages over the adaptive user selection mechanisms, through separation of the auction into demand revelation and sealed-bid phases. We note, however, that our design provides an alternative to their method, with a demand revelation stage for complementary packages (Stage II), and explicit reduction of negative-synergy exposure through a price-vector agent format.

In Stage I of the Schedule Auction, we believe that new market entrants will have a greater opportunity to make simultaneous aggressive bids relative to the SAA, forcing incumbents to either let them into the market or share the burden of keeping them out. Fair entry into the market place has been an expressed consideration in the airport landing slot context, and we feel that the use of the Dynamic Bid Table Auction will also enhance the ability of new entrants to compete in the market by decreasing the opportunity for the "bullying" punishment strategies described by Cramton and Schwartz [**15**] for the SAA. In this behavior observed in actual SAA implementations, an aggressive bidder drives up the price on a competitor as punishment for placing a competing bid. In the Dynamic Bid Table Auction this aggressive behavior is not possible; only price signals are revealed to the bidder at each iteration, making it hard to tell who to bully. Further, each bidder simultaneously competes on several substitutable items without exposure to receiving more than needed; a bullying bidder would have to drive up the price on all items for the punishment strategy to be effective, increasing his chances of winning an item that was bid only to punish a competitor.

Another potential improvement on the approach of available combinatorial auction methods involves the *threshold problem*. The threshold problem is the potential inability of an individual bidder to displace an inefficient package bid without the coordinated efforts of other bidders. Take for example a three bidder auction with four items A, B, C, and D. Suppose that $v_1(\{A, B\}) = v_2(\{C, D\}) = 6$, $v_3(\{A, B, C, D\}) = 9$, and for all other bundles and bidders $v_j(S) = 2\,|S|$. The threshold problem occurs if a bid of 9 on all four items is made by bidder 3 and accepted: neither of the other two bidders is able to submit a package bid on their own that will knock out this bid, but if they were to bid 5 each on $\{A, B\}$ and $\{C, D\}$, respectively, they can clearly overcome the bid of 9 and achieve a more efficient solution.

There is, however, a disincentive to honest revelation in these circumstances known as the *free-rider problem*. If bidder 1 suspects or knows that $v_2(\{C, D\}) = 6$, he may try to only bid 4 on $\{A, B\}$ hoping that bidder 2 will shoulder more of the burden necessary to overcome bidder 3. If bidder 2 takes the same approach, bidding only 4 hoping that bidder 1 will bid 6, the two may fail to overcome the bid of 9 by only bidding a combined 8.

To combat the threshold problem Banks, Ledyard and Porter [**7**] and Kelly and Steinberg [**32**] suggest the use of a publicly available stand-by queue containing voluntarily submitted package/price bids that a bidders may select to combine with their own bids to make winning package bids. Though this technique does address the threshold problem, it is fully open to the difficulties of the free-rider problem. We instead suggest the use of a *third auction stage* to combat both of these problems. Rather than consider collections of non-winning package bids in our own Stage II, our auction format does not address the threshold problem until Stage III. The reasoning for doing this is similar to the reasoning for the separation of Stage I and Stage II: We do not begin to admit package bids in Stage II until after we know such bids will beat the individual item prices derived in Stage I; similarly, we do not want to consider a collection of individual package bids displacing some other collection of package bids until no more improving individual package bids are possible.

In order to combat the threshold problem and the free-rider problem simultaneously, we conduct a sealed-bid auction in Stage III which takes bid information from Stage I and II and attempts to improve upon the allocation found at the end of

Stage II. When a bidder probes a particular set in Stage II, she is told a price that she must pay in order to become the winner of this bundle, at least provisionally. If she chooses to accept the reported price at the time of probing, she becomes the provisional winner of this package. If she decides instead that her value for this bundle is lower than the minimum necessary to become winning unilaterally, she must wait until Stage III to submit her bid on this bundle, hoping that it may then become part of the winning *collection* of bids.

In order to assure participation in Stage II, a bid on a particular bundle in Stage III must be less than would become winning in Stage II. If a bidder (dishonestly) passes on the price of a bundle in Stage II, he forfeits his right to bid this amount in Stage III. Further, every Stage III bundle bid is screened to ensure that it would not have been winning in Stage II; thus an unprobed bid is treated just as a rejected probed bid. For an honest bidder, this rule will not be restrictive when the current (probe) price for a bundle is higher than his true valuation for that bundle. We therefore refer to these restrictions as *honesty constraints*. If a bundle bid is winning at the termination of Stage II, however, the auction has not necessarily discovered the value of the bundle to the bidder. It therefore puts no honesty constraint on a Stage III bundle bid for a bidder that is winning that bundle at the end of Stage II. This provides an additional incentive for participation in Stage II; a bundle won in Stage II may be protected by an arbitrarily large bid in Stage III, despite limitations on the bids of competitors. Within Stage III, the auctioneer will determine an allocation that is more efficient than the one achieved in Stage II (if possible), and compute competitive prices as described in §4.4 and in Chapter 5.

Strategically, this design should make free-riding as difficult as possible. When a bidder decides not to place any further bids on any bundles in Stage II (i.e., when a bidder passes on the opportunity to bid when it is her turn), she relinquishes the ability to bid any amount above the probe price in the final sealed-bid round for any bundle she is not winning at the end of Stage II. This makes it unwise not to accept the probe price except when it is truly too high. Next, faced with the decision of what to report for her valuation of this bundle in Stage III, she cannot test the waters to see how much of a free-ride is possible. Though she may decide to report a valuation that is less than truthful for this bundle, the lower the price she reports the greater the probability that she will regrettably miss an allocation that assigns her this bundle at positive surplus. Honest bidders, on the other hand, have the opportunity to achieve an efficient displacement if one exists, and will pay fair prices determined by the sealed-bid core pricing algorithm described in §4.4 and in Chapter 5.

The three stage auction design we have described has several attractive properties. For the remainder of the chapter we will elaborate upon the theoretical landscape on which this design functions, as well as providing more details into how each stage will proceed. Before delving deeper into this exposition, we list a few of the desirable properties which have been or will be discussed.

Stage I

- Winner determination problem can be solved quickly for a large number of items (on the order of hundreds or thousands of items, an appropriate size for an airport landing-slot auction).
- Bidders may bid competitively on several substitute items at once, without the risk of receiving more than desired.
- Individual item price signals are revealed, diminishing the number of package bids that are necessary in Stages II and III.
- The substitutes property is automatically upheld throughout Stage I.

Stage II

- Bidders may bid on packages without exposure to the risk of paying too much for an incomplete package.
- Each bundle "probe" is computable in polynomial time, as is the auctioneer's validation of a winning bundle/price pair.
- Non-participation in Stage II constrains activity in Stage III, making free-riding and demand reduction risky decisions.

- A bidder may guarantee final possession of the package she is awarded at the end of Stage II, but may be charged to beat out her Stage III competition.

Stage III

- If non-winning bids from Stage II can be combined to overcome a winning package bid, this will be corrected in the final sealed-bid round.
- Competitive prices are determined which share the cost of displacement among those who benefit, alleviating the free-rider problem.

4.3. Package Bidding and Probing

Preferences for complementary bundles introduce a potential for computational difficulty in the auction process. The problems of determining the winning bids and prices for a given round of Stage I can be achieved in polynomial time, as illustrated in Chapter 3. In order to develop Stage II, in which we wish to include package bidding (for complementary packages), we must consider the computational difficulties inherent when complements are introduced.

Consider scenarios in which bidders report demand correspondences (e.g. as in Gul and Stacchetti [**27**] or Ausubel [**2**]). In these auctions each bidder reports which bundles he would be willing to take at current prices (his demand correspondence), after which the prices are changed, and the process repeated until a feasible allocation exists. Both assume the gross substitutes property, but one might wonder if their demand correspondence reporting technique might apply to a more general model of bidders. The following theorem recognizes a difficulty in this scenario: with a slightly general set of bidders not only is the overall problem of finding an efficient allocation $\mathcal{NP}$-hard, finding a feasible allocation for the subproblem at each round is $\mathcal{NP}$-complete.

Theorem 4.1. If the demand correspondence $\{\emptyset, S\}$ is admissible for any $S \subseteq I$, then the problem of determining whether there exists a feasible allocation of items among the bidders according to these reports is $\mathcal{NP}$-complete.

Proof. The problem of allocating items according to demand correspondences is as follows: given a set of demand correspondences $D_1, D_2, ...D_M$, each a set of subsets of the set of all items I, select one subset S_j from each D_j such that $S_j \cap S_{\bar{j}} = \emptyset$ whenever $j \neq \bar{j}$ and $\bigcup_{j \in J} S_j = I$.

It is easy to verify that this problem is in $\mathcal{NP}$. Given a set of S_js we need only verify that $S_j \cap S_{\bar{j}} = \emptyset$ for all $\frac{M(M-1)}{2}$ pairings of distinct bidders and that $\bigcup_{j \in J} S_j = I$. Clearly these verifications can occur in polynomial time.

To show that the problem is $\mathcal{NP}$-complete, we transform a generic set partitioning feasibility problem into an instance of the allocation feasibility problem. Given a set of objects X and a family of subsets of X, $F = \{X_1, X_2, ...X_p\}$, the set partitioning feasibility problem is to find a subset F' of F such that $X_j \cap X_{\bar{j}} = \emptyset$ for any distinct X_j and $X_{\bar{j}}$ in $F'$and $\bigcup_{X_j \in F'} X_j = X$. To transform this into an instance of the demand correspondence allocation feasibility problem simply let $I = X$ and create a bidder j for each X_j where j's demand correspondence $= \{\emptyset, X_j\}$. $\square$

Though several theoretical treatments assume the gross substitutes condition, any extension of bidder preferences which allows for all-or-nothing bids may expect computational difficulties. The following lemma implies that this computational difficulty does not occur when the gross substitutes condition holds, since according to the lemma demand reports of the form $\{\emptyset, S\}$ cannot occur.

Lemma 4.2. If the gross substitutes condition holds, and a particular bidder demands bundle S and the empty set at the current price, then the bidder demands every subset $S' \subseteq S$ at that price.

Proof. We make use of the assumptions that $v_j(\emptyset) = 0$ and $v_j(S) \geq 0$ $\forall S \subseteq I$, and the property that a bidder's net utility function is submodular given that the gross substitutes condition holds [**5**]. Since $D_j(p) \supseteq \{\emptyset, S\}$ at the current price p, we have $u_j(S, p) = u_j(\emptyset, p) = 0 \geq u_j(X, p)$ for any $X \subseteq I$. Submodularity is expressed by $u_j(A, p) + u_j(B, p) \geq u_j(A \cup B, p) + u_j(A \cap B, p)$. Taking $A = S'$ and $B = S \backslash S'$ we have

$$u_j(S', p) + u_j(S \backslash S', p) \geq u_j(S, p) = 0$$

but since $0 \geq u_j(S', p)$ and $0 \geq u_j(S \backslash S', p)$, we must have $u_j(S', p) = 0$ and thus $S' \in D_j(p)$. □

Demand correspondence reporting can present a computational difficulty *at every round* when the gross substitutes condition does not hold. Our overall approach to an auction with complements is therefore as follows: use the bid-table format in Stage I to enforce gross substitutes and reveal price information, and then accept package bids on a basis which avoids the computational burden. The method for avoiding this computational burden (until Stage III) is to only consider "strong" bundle bids in Stage II, those which can immediately become winning without being combined with the bundle bids of others.

Bidders in Stage II will "probe" various sets to determine whether they are willing to place a "strong" bundle bid to compete for this set, or instead to indicate that they have reached an upper bound on what they are willing to pay for this set. For the first bidder in Stage II, this will proceed as described in the following paragraph.

The bidder picks a set of items S which seems most attractive, using the current prices as a first approximation of what bundle is most attractive. When the bidder reports this set, the auctioneer (or an appropriately designed software package) probes it to determine a price. Since no other package bids have been accepted, this task is equivalent to finding a price such that auction efficiency increases if the package bid is accepted and the other items are awarded according to the reports from Stage I. Let z^* be the objective value from P at the end of Stage I, and let z_{-S} be a solution to the primal allocation problem P with the set of restrictions:

$$x_{ijk} = 0 \ \forall i, j, k \text{ with } i \in S$$

The auctioneer may then report to the bidder a price of $p_S = z^* - z_{-S} + 1$. If the bidder is willing to pay this price, then auction efficiency is increased one increment to $z^* + 1$ by awarding S to this bidder at p_S and awarding all other items according to the assignment solution yielding z_{-S}. Finding this assignment and corresponding value of z_{-S} is not computationally burdensome: we merely solve a smaller assignment problem than was solved at each round of Stage I.

In order to continue this procedure past the first bidder in Stage II, we must first define the set of packages for which package bids have been provisionally accepted, $\mathcal{S}$. Now, when a bidder probes set S, the auctioneer takes the current auction value z^*, and computes z_{-S} a solution to the primal allocation problem P (from §3.2) with the set of restrictions:

$$x_{ijk} = 0 \ \forall i, j, k \text{ with } i \in S \text{ or } i \in T \text{ where } T \in \mathcal{S} \text{ with } S \cap T = \emptyset$$

The auctioneer reports a value $p_S = z^* - z_{-S} + 1$ as before. If this price is accepted, the new bid will push any overlapping bids out of the current allocation, so that any items in a pushed out package bid but not in S must be allocated according to the assignment information from Stage I. The bidder will pay the amount necessary to compensate the auction for the drop in efficiency from displacing any currently accepted package bids and reverting to bid table values for leftover items. Again the computations are easily handled as restricted versions of the assignment problem P. Since all such restrictions are simply bounds on the variables of P, the total unimodularity of problem P, and hence polynomial solvability is not sacrificed.

In the Stage II package auction (as described until this point) there is a potential exposure to inefficiency. A bidder may win a particularly desirable package in Stage II but then also receive several other individual items from the collection of items that were not awarded as part of a package. This will occur, for example, if the bidder intends a slot in his package bid for a particular flight and then wins a slot in the column corresponding to same flight in his bid table. This problem easily disappears if we allow (or force) bidders to specify which of their agents (bid table columns, flights) they plan to use the package of slots to satisfy, and simply removing these columns from consideration in the calculation of z_{-S}. This *Package/Agent Designation* suggests a reasonable user-interface for Stage II probing; the bundle to be probed may be specified by simply clicking on the bid table entry corresponding to the row of the desired slot and the column of corresponding agent (flight) for each slot in the bundle. The auctioneer removes the corresponding columns from the bidder's bid table until (perhaps) the bundle bid is no longer winning. This assures that a bidder does not receive multiple slots for the same flight, even when package bids are considered.

This removal of bid table columns suggests a further incentive to participation in the Dynamic Bid Table Auction of Stage I. As more packages are accepted into the final allocation, more and more bid table columns are removed. Further, certain slots may remain "loose," not being incorporated in any package bid. As fewer bid table columns remain active in the auction, competition on these loose slots diminishes, resulting in lower final prices for slots that are won from bid table entries. Intuitively, a bid table entry is very flexible and may be applied at any time; by making this flexible offer to pay for an item in the early rounds a bidder may be rewarded with a discounted price in the final allocation.

4.4. The Threshold Problem, the Free-Rider Problem and Efficiency

The auction format described in the present work differs from earlier combinatorial auction literature in its treatment of the threshold problem (as described in §4.2). As mentioned above, some advocate a publicly available stand-by queue of non-winning bids which an interested bidder may combine with her own to form a winning bid (see [**7**][**32**]). We argue that this revelation of information is beyond what is necessary to achieve efficiency and allows for tactical manipulation with little if any benefit. As previously mentioned, this tactical manipulation comes in the form of free-riding. A bidder with the ability to view the valuations of others will find it in her best interests to reveal as little valuation as is necessary to form a winning bid, placing as much burden as possible on those who have revealed their valuations to the stand-by queue. Intelligent bidders may therefore cautiously shade down their own revelation to the queue, possibly not revealing enough information for an efficient allocation to be reached. Although we do not analyze this situation with rigorous game-theory, it should be clear that the presence of the stand-by queue introduces two potential problems associated with a disincentive to truthful revelation: bidders taking advantage of the revealed information of others (free-riding) and the inability to achieve efficiency due to cautious shading/free-riding defense strategies.

Even if for some altruistic reason we were to assume truthful revelation (thereby nullifying the free-rider problem) another problem remains with the stand-by queue approach. Maintaining the availability of this bid information throughout the auction seems to diagnose the threshold problem as a barrier to bidding throughout the auction, when in fact the threshold problem should be recognized as a barrier to efficiency only in the final allocation. A bidder in an auction with a stand-by queue may spend valuable computation time attempting to find a collection of available queue bids to combine with his own, only to have the successfully combined coalitional bid knocked out in subsequent rounds. To clarify the situation, we advocate a new definition of the threshold problem:

Definition: The *Threshold Problem* occurs when a collection of bids, each of which cannot become winning unilaterally, can be combined to reach an efficient final allocation.

There are two changes here from the definition given loosely in §4.2 (and in the literature at large.) One distinction is the word "final" which makes it unnecessary in particular to consider a set of bids which may be combined to displace a bid that eventually is not winning (as is the case in the intermediate phases of AUSM [**7**] or PAUSE [**32**].) A second change is "each of which cannot become winning unilaterally." If a bidder is willing to submit a bid which can become winning unilaterally, we would want her to do so, rather than hiding this high valuation only to submit this into the final sealed-bid phase when it is too late for her opponents to react. To encourage revelation during Stage II we assume that a bidder has bid truthfully and hold her to it in Stage III via bounds on her Stage III bids. These "honesty constraints" make it risky to hide preferences in Stage II, as doing so will limit your ability to bid aggressively in Stage III.

These observations suggest that the threshold problem should not be addressed until a final sealed-bid auction, regardless of what type of sealed-bid auction is used. For the remainder of this section we describe the specific winner determination problem for a sealed-bid auction tailored to the multi-stage auction as developed to this point. In particular, we want to maintain the bid table entries as bid information, allow for package bidding, and uphold the flight designations for each package bid as described in §4.3. We now describe an integer programming formulation for determining winners, treating all revealed bid information as binding.

In addition to each (final) bid table entry b_{ijk}, we will have (from Stage III) some indexed list of L package bids $\{(B_1, S_1), ...(B_l, S_l), ...(B_L, S_L)\}$, where each (B_l, S_l) represents a bid of B_l for set S_l. This set of package bids will contain any bid that is winning at the end of Stage II, and any other bids submitted for Stage III after having been screened to ensure that each B_l could not become winning at the end of Stage II. There is a natural OR relationship among the package bids of the same bidder. It may be helpful to allow bidder the ability to express an XOR relationship among a particular collection of bids, but since this may be handled theoretically by the introduction of dummy items without changing the formulation of P_3 below, we currently ignore this consideration for the sake of simplicity.

We introduce the notation A_l to denote the set of agents that a bidder has associated with her particular package bid (B_l, S_l), and 0-1 decision variable y_l which equals one if and only if package bid (B_l, S_l) is accepted. In the following formulation we will wish to express that if (B_l, S_l) is a winning package bid then the bid table columns associated with tasks in A_l will become inactive (i.e. not win any slots). To find an optimal allocation we solve the following allocation problem for Stage III, which we denote P_3:

$$
\begin{aligned}
(\mathrm{P}_3) \qquad & \max \sum_{(i,j,k)\in I\times J\times A_j} b_{ijk}\cdot x_{ijk} + \sum_{l=1 \text{ to } L} B_l \cdot y_l \\
\text{subject to } & \sum_{i\in I} x_{ijk} + \sum_{l|(j,k)\in A_l} y_l \le 1, \ \forall (j,k) \text{ with } j\in J \text{ and } k \in A_j \\
& \sum_{j\in J}\sum_{k\in A_j} x_{ijk} + \sum_{l|i\in S_l} y_l \le 1, \ \forall i \in I \\
& x_{ijk} \in \{0,1\}, \ \forall i,j,k \\
& y_l \in \{0,1\}, \ \forall l
\end{aligned}
$$

This modification of formulation P (from §3.2) introduces the y_l variables placed so that if a package bid wins its specified collection of items then the tasks associated with this bid will not win other items, and no other bid which includes any of those items may be accepted.

The solution to P_3 gives the final efficient winners and their awarded bundles, but does not tell us what prices to charge the bidders for the items they receive. Pay-as-bid is one extreme possibility, but clearly this pricing rule provides the strongest incentives for bid shading or reduction, risking a less efficient outcome. The other extreme possibility is to adopt a VCG payment structure which is guaranteed to be incentive compatible. There are, however, several problems with VCG payments when superadditive valuations are present, as noted earlier. We instead terminate our auction with a core outcome as advocated (for example) in the recent work of Ausubel and Milgrom [**5**] and Hoffman et al [**29**]. The process we develop for achieving these favorable outcomes is treated separately in Chapter 5, but before doing so we explore the ideas of this chapter in greater depth by working through a sample auction.

4.5. Example of a Schedule Auction for Airport Landing Slots

In this section we demonstrate the three-stage Schedule Auction with an illustrative example. We assume a small auction environment for the sake of exposition, but one that is large enough to show some of the intricacies of the three-stage format. We describe an auction with 5 airlines bidding for 10 airport landing slots. For each of these bidding airlines, we introduce a sample strategy based on our general assessments of the landing-slot application literature ([**6**],[**21**]) and a recent workshop [**49**] on auctioning landing-slot rights. We work through the stages of the auction, simulating participation with bidders behaving in a straightforward manner, as described below. Before introducing the simulated bidding strategies in detail, we discuss a few simple procedural measures to encourage participation in Stage I and to avoid a monopoly situation.

To encourage participation in the Dynamic Bid Table Auction of Stage I, we allow the auctioneer to make Package/Agent Designation (described in §4.3) mandatory in Stages II and III, and further stipulate that each Package/Agent Designation must be composed of an assignment of the items in the package to agents using only positive bid table entries. Thus, in Stages II and III a bidder can specify his package of interest by highlighting positive bid entries in his bid table, with only one entry per row and only one entry per column selected. This mandatory Package/Agent Designation requires a bidder to keep at least one positive bid table entry in the row of any item he may be interested in for the remainder of the auction. Further, if he is interested in maintaining eligibility to win a package of size n, he must keep at least one positive bid in at least n columns (i.e., for at least n agents). We note that if a bidder decides to simply keep a single positive entry for each item, each with its own dedicated agent, that he is essentially participating in a Simultaneous Ascending Auction. The strength, however, of the bid table format is that the bidder does not necessarily need to dedicate a single agent (column) to each item (row), but may instead reduce his negative-synergy exposure by placing some bids in the same column.

We note finally that this mandatory Package/Agent may be made even more stringent (thus encouraging more revelation in Stage I) by replacing the requirement of a positive bid entry with a requirement that each designated entry be above a reservation value, perhaps one that is determined dynamically over the course of the auction. This may take the form of a constraint saying that bid table entries less than half of the final Stage I price for that row are ineligible for package designation, for example. The auctioneer should be careful however that efficiency is not sacrificed with too stringent a restriction, which could limit the behavior of a bidder with very low individual item values but high positive synergy. In practice, this may be avoided with an estimated upper bound on positive synergy, but for the example of this section, we simply assume the weakest form of this restriction, that a bid table entry must be positive to remain eligible for Package/Agent designation.

With mandatory Package/Agent Designation in place, any easy way to enforce bundle-size limitations quickly presents itself. Since a bidder may not receive any more items than he has bid table columns containing positive entries, the maximum bundle size is effectively constrained by the total number of bid table columns allowed by the auctioneer. In the example of this section, the auctioneer stipulates that no bidder may win more than 5 of the 10 available landing slots, and hence each bid table can have only 5 columns.

We now describe the strategies of five simulated airlines, V, W, X, Y, and Z. The items for auction are 10 landing slots at a particular airport between the times of 1:00 PM and 3:15 PM on Fridays throughout an entire season. Because no such auction has yet taken place (leaving us with no historical estimates of value), the monetary units in our example are arbitrary. The final Stage I bid tables for these airlines are collected in Figure 4.3 on page 116. We now describe the strategies of each of our five fictional airlines.

Airline V: This airline represents a small carrier with simple preferences, that is most interested in just buying its way into this market with one or two slots. As a new entrant, this airline is unconcerned with which slots it receives, but would be willing to pay at most 124 for one and most 224 for two slots. With these simple preferences *Airline V* has no need for package bidding; its participation is complete with the submission of its final bid table in Stage I (see Figure 4.3). Since this airline is only interested in packages of two or fewer items, we see that only two of its bid table columns contain positive entries.

Airline W: Similar to *Airline V*, *Airline W* is a small new entrant interested in two or fewer time slots, and does not perceive any positive synergy among the items it wins. This airline is, however, specifically interested in 1:00 or 1:15 landing slots and experiences slowly accumulating costs of delay for later slots. These accumulating costs of delay are evidenced by the decreasing columns of *Airline W*'s bid table.

Airline X: This medium-sized airline is interested in purchasing at most four slots to maintain the four flights it has historically landed in the 1:00-3:15 time frame. It sets a maximum price for each slot in the historical landing time of each of the

Figure 4.3. An auction for airport time slots: Stage I. Winning entries appear in bold, and prices are given by the displayed vector p

Airline V

	1	2	3	4	5
1:00	124	100			
1:15	124	100			
1:30	124	100			
1:45	124	100			
2:00	124	100			
2:15	124	100			
2:30	**124**	100			
2:45	124	100			
3:00	124	100			
3:15	124	100			

Airline W

	1	2	3	4	5
1:00	**128**	108			
1:15	128	108			
1:30	128	107			
1:45	128	107			
2:00	128	107			
2:15	125	105			
2:30	124	104			
2:45	113	93			
3:00	113	93			
3:15	110	91			

Airline X

	1	2	3	4	5
1:00					
1:15					
1:30	115				
1:45	103	110			
2:00	90	110	**110**		
2:15	90	100	107		
2:30		95	90		
2:45			85	**112**	
3:00				111	
3:15				90	

Airline Y

	1	2	3	4	5
1:00	107	107	107	87	87
1:15	114	**114**	114	94	94
1:30	120	120	**120**	100	100
1:45	**114**	114	114	94	94
2:00	107	107	107	87	87
2:15					
2:30					
2:45					
3:00					
3:15					

Airline Z

	1	2	3	4	5
1:00	114	114	114		
1:15	113	113	113		
1:30					
1:45					
2:00	114	114	114		
2:15	113	113	**113**		
2:30					
2:45					
3:00	**114**	114	114		
3:15	113	**113**	113		

$p =$

1:00	108
1:15	108
1:30	115
1:45	108
2:00	108
2:15	107
2:30	104
2:45	104
3:00	104
3:15	104

corresponding flights, with decreasing bid table entries below, again reflecting accumulating costs of delay. Unlike *Airline W*, *Airline X* will not settle for a slot that is an hour or more after the targeted time slot for each flight, perhaps because this would be too disruptive to its existing schedule or operational strategy. These preferences are expressed with decreasing columns of at most four consecutive positive entries in *Airline X*'s bid table. Further, based on previous experience, *Airline X* calculates that it can save 21 monetary units in operational costs if three consecutive slots are purchased. This may reflect, for example, an advantage in the scheduling of ground crew, who would not sit idle if servicing consecutive slots. This expression of positive synergy (21 units) cannot be conveyed in a bid table, but will help determine *Airline X*'s package bids in Stages II and III.

Airline Y: This large incumbent airline has an established "banking" strategy in which it lands several airplanes as close to 1:30 as possible, so that it may reorganize its passengers for a round of outgoing flights taking off around 3:30 (after the slots of this auction). It decides that slots after 2:00 arrive too late for outgoing passengers on 3:30 flights, and so does not place any positive bids on the later slots. *Airline Y* forecasts diminished revenue on the fourth and fifth flights landed in this time block, but experiences economies-of-scale: 35 monetary units in operational savings for any four flights landed in the 1:00 to 2:00 range, with an additional 1 unit of value if the four slots are consecutive. Again, the diminished returns can be expressed with reductions of the entries in the fourth and fifth columns of *Airline Y*'s bid table, while the positive-synergy bonus (from the economies-of-scale) must wait until Stages II and III to be expressed.

Airline Z: *Airline Z* is medium-sized (demanding at most three slots) and has a "shuttling" strategy. It would like to land one airplane every hour during this time period, on the hour or just after. They calculate 34 additional monetary units of benefit from any bundle giving them the ability to offer a reliable shuttle service with three hourly landings during this time period. Further, it predicts greater ease in advertising this service with landings specifically at 1:00 and 2:00, and perceive 3 additional units of synergy for any bundle allowing for the shuttle service and containing both of these slots. This suggests several complementary bundles for this bidder, with positive synergy that cannot be expressed in a bid table. Still, according to the mandatory Package/Agent Designation, *Airline Z* must keep positive bid table entries for all time slots of possible interest, while simultaneously keeping positive bid entries in at three least columns to remain eligible for bundles of size three.

Given the preferences described here, we now begin to run through the steps of the three-stage Schedule Auction for this illustrative example. Each round of Stage I proceeds with bidding airlines each submitting newly adjusted bid tables based on the lowest Walrasian equilibrium prices announced by the auctioneer. It is feasible that an airline knows what it wants its final entries to be and simply submits its final bid table at the outset, but in general learning about the market prices will guide the bidders through the process of writing their final bid table. We therefore present only the final bid tables in Figure 4.3.

Although we do not present every step of Stage I for the sake of brevity, we note that important learning may take place over the course of Stage I, providing valuable information to the bidders and affecting their final bid tables. For example, *Airline V* may have begun the auction with preferences for more than two landing slots, but settled on two or fewer after the prices reached a certain level. At that point they would stop bidding in their third, fourth, or fifth columns and allow those entries to become inactive. For simplicity, however, we present these final bid tables as if each airline had complete knowledge of its preferences and filled out its final bid table accordingly. Thus, in order to focus on the flow of the entire Schedule Auction, let us assume that Stage I has terminated with the final bid tables as given in Figure 4.3.

Comparing the bold winning entries in Figure 4.3 to the prices for the corresponding items given by the vector p, we see that every item is awarded with positive surplus (i.e., the price is always lower than the value of the winning entry). Each bidding airline may compute its total current surplus (payments minus bids) to use as a guide for bidding in Stage II. In order to model a "myopic best response" strategy for each bidder in Stage II, we assume that at each round in Stage II a bidder compares the surplus for a package bid outcome at the probed price, to the surplus of the currently proposed outcome. A bidder then accepts the probed package bid outcome that provides the most surplus, whenever this surplus is greater than that of the currently proposed outcome. Initially, we follow this myopic best response scenario through to the conclusion of the auction, and afterward compare the results to the outcome obtained under a more "forthcoming" strategy, described below.

In order to compute surplus at each round of probing, each airline must compute its value (maximal bid) on various bundles. To begin Stage II, let us assume that each airline chooses some bundles of interest to bid on as packages, or at least to probe for a price quote. Since *Airline V* and *Airline W* do not have positive-synergy preferences for bundles, we focus our attention to *Airlines* X, Y, and Z, who compute the following values for bundles, based on the preference descriptions given above.

$$v_X(\{1{:}30,\ 1{:}45,\ 2{:}00\} = 356 \qquad v_Y(\{1{:}00,\ 1{:}15,\ 1{:}30,\ 1{:}45\} = 471$$

$$v_X(\{1{:}45,\ 2{:}00,\ 2{:}15\} = 341 \qquad v_Y(\{1{:}00,\ 1{:}15,\ 1{:}30,\ 2{:}00\} = 463$$

$$v_X(\{2{:}00,\ 2{:}15,\ 2{:}30\} = 311 \qquad v_Y(\{1{:}00,\ 1{:}15,\ 1{:}45,\ 2{:}00\} = 457$$

$$v_X(\{2{:}15,\ 2{:}30,\ 2{:}45\} = 323 \qquad v_Y(\{1{:}00,\ 1{:}30,\ 1{:}45,\ 2{:}00\} = 463$$

$$v_X(\{2{:}30,\ 2{:}45,\ 3{:}00\} = 312 \qquad v_Y(\{1{:}15,\ 1{:}30,\ 1{:}45,\ 2{:}00\} = 471$$

$v_Z(\{1{:}00,\ 2{:}00,\ 3{:}00\} = 379 \qquad v_Z(\{1{:}15,\ 2{:}00,\ 3{:}00\} = 375$

$v_Z(\{1{:}00,\ 2{:}00,\ 3{:}15\} = 378 \qquad v_Z(\{1{:}15,\ 2{:}00,\ 3{:}15\} = 374$

$v_Z(\{1{:}00,\ 2{:}15,\ 3{:}00\} = 375 \qquad v_Z(\{1{:}15,\ 2{:}15,\ 3{:}00\} = 374$

$v_Z(\{1{:}00,\ 2{:}15,\ 3{:}15\} = 374 \qquad v_Z(\{1{:}15,\ 2{:}15,\ 3{:}15\} = 373$

Under the assumption of myopic best response, we begin Stage II with *Airline X* probing its bundles of interest to determine the current price for each. To do so, the airline must designate a set of agents (flights) to be used with each package so that the corresponding columns may be removed from its bid table. Throughout all treatments in this section, we assume that each bidder designates agents to packages according to a maximum value assignment of the package to their bid table. The results of *Airline X*'s probes in the first round are as follows.

Package	Value	Price	Other items awarded	Value	Price	Surplus
{1:30, 1:45, 2:00}	356	346	{2:45}	112	104	18
{1:45, 2:00, 2:15}	341	339	{2:45}	112	110	4
{2:00, 2:15, 2:30}	311	337	∅			-26
{2:15, 2:30, 2:45}	323	330	∅			-7
{2:30, 2:45, 3:00}	312	337	∅			-25

We see that since some of *Airline* X's bid table columns remain active he may receive more items than just those in the package of interest, even when some columns are removed to accommodate the package bid. We see that according to the probes only two of these bundles would yield positive surplus for *Airline* X, and only one would yield more surplus than the outcome at the termination of Stage I, which provided *Airline* X with 10 units of surplus. *Airline* X therefore accepts the probed price for the bundle {1:30, 1:45, 2:00}, which offers a better outcome than the Stage I outcome in terms of surplus. Once a probed price/package is accepted, *Airline* X's turn is done, and the auction moves to the next randomly selected bidder's turn to probe. We assume that this random selection never gives the same bidder two turns in a row, and includes all bidders in each round.

The collected results for Stage II of this auction (under the myopic best response assumption) are tabulated in Figure 4.4. We assume that at each randomized round-robin turn each airline probes each bundle of interest, and accepts the outcome with the highest surplus (for that airline) as long as that outcome provides more surplus than the current outcome (i.e., the one determined by the last accepted probe). We show only the accepted package bid (if any) at each turn and give the new surplus computed for *Airlines* X, Y, and Z after the acceptance and reallocation, denoted s_X, s_Y, and s_Z, respectively.

Figure 4.4. Bundle probing in Stage II, assuming myopic best response

	Bidder	Accepted Bundle	Price	s_X	s_Y	s_Z
				10	17	25
Round 1	Z	{1:00, 2:00, 3:00}	348	14	15	31
	X	{1:30, 1:45, 2:00}	346	18	6	27
	Y	{1:00, 1:15, 1:30, 1:45}	461	6	10	21
Round 2	Z	{1:00, 2:00, 3:00}	351	14	15	28
	X	{1:30, 1:45, 2:00}	349	15	6	27
	Y	{1:00, 1:15, 1:30, 1:45}	464	6	7	21
Round 3	X	{1:30, 1:45, 2:00}	351	13	6	27
	Z	{1:00, 2:15, 3:00}	345	17	6	30
	Y	None accepted				
Round 4	X	None accepted				
	Z	None accepted				

Along the way, we observe some of the beneficial demand revelation properties of Stage II with a randomized round-robin bidding pattern. Particularly, since we assume in this treatment that the only information bidders receive at each turn is the price for probed bundles (i.e., they do not see what others win in their turns), it is difficult for bidders to coordinate their bids to not interfere with one another. For example, *Airline* X is only ever interested in its favorite package, {1:30, 1:45, 2:00}, for the entirety of the auction, because no other package ever maximizes surplus at the probed prices. In Rounds 1 and 2, the randomized ordering places *Airline* Z before *Airline* X, making it impossible for *Airline* Z to choose a non-interfering bundle through probe prices alone. In Round 3, however, the randomization puts *Airline* Z after *Airline* X, so that his Round 3 probes show a lower price for bundles which do not conflict with *Airline* X. *Airline* Z is therefore able to achieve more surplus from the bundle {1:00, 2:15, 3:00} than from its favorite bundle, {1:00, 2:00, 3:00}, because it does not knock out the bid of *Airline X*. We emphasize that through randomized round-robin probing, *Airlines X* and *Z* can achieve this coordination without ever seeing what the other is bidding on. We see that Stage II terminates after less than four full rounds of probing, with *Airline X* winning a package of consecutive flights, {1:30, 1:45, 2:00}, and *Airline Z* winning a package supporting his shuttle service, {1:00, 2:15, 3:00}.

Under the myopic best response assumption, *Airline Y* ends up not accepting a probed price offering positive surplus in Round 3, because the amount of this positive surplus is less than the surplus of the currently proposed solution. Consequently, several "honesty constraints" take effect in Stage III, limiting the maximum amount that airlines may bid in the final sealed-bid stage. Consider the following table of bundle probes for *Airline Y* in Round 3 of Stage II:

Package	Value	Price	Surplus
{1:00, 1:15, 1:30, 1:45}	471	467	4
{1:00, 1:15, 1:30, 2:00}	463	467	-4
{1:00, 1:15, 1:45, 2:00}	457	467	-10
{1:00, 1:30, 1:45, 2:00}	463	467	-4
{1:15, 1:30, 1:45, 2:00}	471	466	5

We see that two packages offer positive surplus, but neither offers more surplus than the currently proposed outcome (following Z's bid in Round 3) with surplus of 6. *Airline* Y therefore does not accept any probed package bids in Round 3, in accordance with our assumption of myopic best response.

Because of this non-acceptance, the honesty constraints dictate that no bundle bid in Stage III may exceed the probed price for that bundle. *Airline* Y may therefore not submit a bid higher than 467 on the bundle {1:00, 1:15, 1:30, 1:45}, nor a bid higher than 466 on the bundle {1:15, 1:30, 1:45, 2:00}. Indeed, according to the honesty constraints *Airline* Y may not bid higher than 467 on the bundle {1:00, 1:15, 1:45, 2:00}, for example, but since this bound exceeds *Airline* Y's valuation for this bundle, the constraint will not be restrictive.

Figure 4.5 shows the results of the entire auction at each stage, assuming myopic best response in Stage II. We assume that each bidder bids honestly on bundles in Stage III, except where this would violate the honesty constraints. The general method used to compute the final (bidder-Pareto-optimal core) payments is given in Chapter 5. This method assures that the payment mechanism corresponds to the VCG outcome whenever that outcome is in the core, and takes the minimal-payment core outcome closest to the VCG outcome otherwise. This approximation of the VCG outcome reinforces our assumption of honest submission of package values in Stage III. For the results of this section, the algorithm does terminate at the VCG

Figure 4.5. Results for all Stages, assuming myopic best response in Stage II

Stage I	Bidder	Bundle	Value	Payment	Surplus
	V	{2:30}	124	104	20
	W	{1:00}	128	108	20
	X	{2:00, 2:45}	222	212	10
	Y	{1:15, 1:30, 1:45}	348	331	17
	Z	{2:15, 3:00, 3:15}	340	315	25
totals			1162	1070	92

Stage II	Bidder	Bundle	Value	Payment	Surplus
	V	{3:15}	124	91	33
	W	{2:30}	124	100	24
	X	{1:30, 1:45, 2:00, 2:45}	468	451	17
	Y	{1:15}	114	108	6
	Z	{1:00, 2:15, 3:00}	375	345	30
totals			1205	1095	110

Stage III	Bidder	Bundle	Value	Payment	Surplus
	V	{3:00}	124	102	22
	W	{2:30}	124	102	22
	X	{2:45}	112	100	12
	Y	{1:00, 1:30, 1:45, 2:00}	463	460	3
	Z	{1:15, 2:15, 3:15}	373	321	52
totals			1196	1085	111

outcome, since it does happen to be in the core, allowing the avid reader to verify the results of this section on her own, with no need to skip ahead to the more advanced technique described in Chapter 5.

Observing Figure 4.5, we see that total value (for all bidders) increases from the end of Stage I to the end of Stage II, but drops from the end of Stage II to the Stage III final outcome. This forces to ask, why would the auction pick an outcome with lower total value given more bid information in Stage III? The answer is that due to the nature of probe pricing in Stage II and honesty constraints in Stage III, some of the values for various bundles remain hidden from the auctioneer. For example, in Stage II the auctioneer cannot be sure that any bidder values a particular bundle more than the price they have accepted for that bundle. The auctioneer could therefore only know that *Airline X*'s value for its awarded bundle at the end of Stage III is at least 463 (the price of the bundle bid, 351, plus the known value from the bid table entry of 112), but cannot tell that *Airline X* actually values the bundle at 468. Similarly, because of honesty constraints *Airline Z* can bid at most 364 for its awarded bundle in Stage III, so some of *Airline Z*'s value for this bundle (373) remains hidden from the auctioneer. We therefore compute the total *perceived* value to be 1162, 1170, and 1187, for Stages I, II and III, respectively. Thus as far as the auctioneer can tell, perceived value (as opposed to *actual* value as shown in Figure 4.5) is increasing from each stage to the next.

We see also (in the instance of Figure 4.5) that total payments and surplus are increasing from the end of Stage I to the end of Stage II. Indeed, payments must increase from the Stage I outcome to the Stage II outcome; they increase by one increment for every package bid accepted. (We do not discuss increment size for Stage II here, though it may be fined tuned in practice to speed up or slow down the pace of the auction.) It should be remembered, however, that payments in Stages I and II are merely price signals, and that final payments may be higher or lower than as signaled in earlier stages. Payments may seem to rise from Stage II to III since more demand and hence more price competition has been revealed. This is the case for small entrant *Airlines V* and W, who saw good prices on a single item rise in Stage III. *Airline Z* provides an example of the opposite possibility; the price signals from Stage II (which measure a minimum *unilateral* payment) can overestimate the payment that is needed by the end of the auction, especially if the package bid can be coordinated with those of others. Overall, in this instance total payments dropped from Stage II to Stage III, though our experience with other examples (not discussed in this text) show that this may not always be the case.

The Schedule Auction is designed with honesty constraints to encourage demand revelation in Stage II, and a bidder-Pareto-optimal pricing rule in Stage III to minimize the costs of this demand revelation (in terms of increased payments) as much as possible. Do the honesty constraints work? Would it have been a better idea for the bidders to play a more forthcoming strategy (revealing more demand) in Stage II? We now demonstrate the possibility that no bidder is made worse off if the myopic best response strategy is replaced by a more truthful strategy.

To demonstrate this interesting possibility, let us return to the end of Stage I (i.e., the bids of Figure 4.3) and re-run the auction with the assumption of a more forthcoming set of bidders in Stage II. In particular, we say that a "forthcoming" bidder accepts the probed package bid outcome that provides him the most surplus, as long as the price of the corresponding bundle is less than or equal to his value for that bundle (and he is not already winning that bundle from the previous round). For the following treatment we assume that all bidders are forthcoming.

With all airlines accepting bundles in this forthcoming manner at each round of probing, the results of Stage II are given in Figure 4.6. Because the results for the first two rounds are the same as for myopic best response bidders (as in Figure 4.4), we begin Figure 4.6 at Round 3. In particular, we see the first action that violates myopic best response occurs at *Airline Y*'s turn in Round 3. At this point in the previous treatment, *Airline Y* refuses all probed prices because each one delivers less surplus than in the previously proposed outcome. In Round 3 of Figure 4.6, however, we see that a more forthcoming *Airline Y* chooses to accept a bundle bid that causes a decrease in its personal surplus (from 6 to 5). Though this may seem to be a bad move, the well-informed *Airline Y* knows that that information returned by the probed prices is merely signalling information, and that it is more important to leave open the possibility of higher bids in Stage III.

Watching the rest of Stage II unfold under the assumption of forthcoming bidders, we see that *Airline Z* has found a bundle that the other bidders never find it in their best interest to knock out. *Airline Z* therefore sits complacently while *Airlines X* and *Y* vie for the opportunity to place a complementary package bid. For the remainder of Stage II, these two bidders continue to knock each other out of

Figure 4.6. Bundle probing in Stage II, assuming forthcoming bidders

	Bidder	Accepted Bundle	Price	s_X	s_Y	s_Z
Round 3	X	{1:30, 1:45, 2:00}	351	13	6	27
	Z	{1:00, 2:15, 3:00}	345	17	6	30
	Y	{1:15, 1:30, 1:45, 2:00}	466	12	5	30
Round 4	X	{1:30, 1:45, 2:00}	353	15	6	30
	Z	None accepted				
	Y	{1:15, 1:30, 1:45, 2:00}	468	12	3	30
Round 5	Z	None accepted				
	X	{1:30, 1:45, 2:00}	355	13	6	30
	Y	{1:15, 1:30, 1:45, 2:00}	470	12	1	30
Round 6	X	None accepted				
	Y	None accepted				
	Z	None accepted				

position until *Airline X* finds that the probed price is greater than its value for any bundle and quits. (Note that some of the surplus seen in Figure 4.6 for *Airline X* comes from the 2:45 slot which is won from the bid table entries at positive surplus.) We see that at each round in Figure 4.6, the probed price for each of these two bidders increases by two increments, one to match the previous bid of the competitor and a second increment to raise the stakes, calling to mind a game of poker.

Looking at the final results under the assumption of forthcoming bidders in Figure 4.7, we see that enough information is revealed in Stage II to achieve the Stage III allocation (though this is not always the case). Further, it can be verified that this is the efficient allocation. Under the forthcoming strategy in this example it turns out that the honesty constraints on *Airline Z*'s Stage III bids did not affect the outcome of the auction in terms of efficiency. For example, according to his final Stage II

Figure 4.7. Results for all Stages, assuming forthcoming bidders in Stage II

Stage I	Bidder	Bundle	Value	Payment	Surplus
	V	{2:30}	124	104	20
	W	{1:00}	128	108	20
	X	{2:00, 2:45}	222	212	10
	Y	{1:15, 1:30, 1:45}	348	331	17
	Z	{2:15, 3:00, 3:15}	340	315	25
totals			1162	1070	92

Stage II	Bidder	Bundle	Value	Payment	Surplus
	V	{3:15}	124	91	33
	W	{2:30}	124	100	24
	X	{2:45}	112	100	12
	Y	{1:15, 1:30, 1:45, 2:00}	471	470	1
	Z	{1:00, 2:15, 3:00}	375	345	30
totals			1206	1106	100

Stage III	Bidder	Bundle	Value	Payment	Surplus
	V	{3:15}	124	91	33
	W	{2:30}	124	100	24
	X	{2:45}	112	100	12
	Y	{1:15, 1:30, 1:45, 2:00}	471	464	7
	Z	{1:00, 2:15, 3:00}	375	317	58
totals			1206	1072	134

probes, *Airline Z* may not bid more than 361 for the bundle {1:00, 2:00, 3:00}, even though its true value for this bundle is 379. *Airline Z* was therefore able to hide some of his value for bundles (even from the auctioneer) without consequence, a beneficial privacy-preservation property. More importantly, comparing Figure 4.7 with Figure 4.5, we see that no bidder strictly prefers the myopic best response outcome to the forthcoming outcome, with only *Airline X* being indifferent between the two. This comparison illustrates the strength of honesty constraints for encouraging demand revelation for complementary bundles.

With this example we conclude our discussion of Schedule Auctions, and move on to discuss the calculation of bidder-Pareto-optimal core prices performed in Stage III. We treat this new computational technique separately in Chapter 5, since it may be applied in a more general combinatorial auction setting.

CHAPTER 5

Generation and Selection of Core Outcomes

Definition: Let an *outcome*, Γ, refer to an allocation and set of payments for bidders in a combinatorial auction. Let the *coalition*, C_Γ, refer to the set of bidders receiving items under outcome Γ.

Perhaps the most obvious problem with VCG payments in terms of ex-post satisfaction is that the VCG outcome may not be a "core outcome". The core conditions for a one-sided (forward) auction may be stated by the following definitions, where bidder j *weakly prefers* outcome Γ_1 to Γ_2 if Γ_1 provides bidder j with utility greater than or equal to the utility of outcome Γ_2.

Definition: An outcome Γ is *blocked* if there is an alternative outcome Γ_B which generates strictly greater revenue for the seller and for which every bidder in C_{Γ_B} weakly prefers Γ_B to Γ. C_{Γ_B} may be referred to as a *blocking coalition*.

Definition: An outcome Γ that is not blocked is called a *core outcome*.

It is well known that a VCG outcome may not be a core outcome. Consider the following three-bidder, two-item example from Ausubel [**2**]. Let $b_1(\{A, B\}) = b_2(\{A\}) = b_3(\{B\}) = 2$. In the efficient allocation, bidder 2 wins item A while bidder 3 wins item B. VCG payments for each winning bidder are easily computed to be zero. This is not a core outcome because bidder 1 would prefer to pay any amount up to 2 units to receive both items, an outcome which is clearly more desirable to the seller. If instead bidders 2 and 3 each pay 1, the coalition containing just bidder 1 no longer blocks, and the outcome is in the core.

As an alternative to VCG pricing, Ausubel and Milgrom [**5**] suggest an *ascending proxy auction* which arrives at a more desirable core outcome. DeVries, Schummer and Vohra [**17**] interpret this algorithm for winner/payment determination as a subgradient algorithm: at each moment in the process the algorithm determines a revenue maximizing allocation given some current set of prices, and a net utility maximizing bundle for each bidder. Each non-anonymous bundle price is then adjusted up by one increment for each bundle that is not contained in the "seller's choice" allocation. This process corresponds to incremental stepping in the direction of a subgradient. As is commonly the case with the subgradient algorithm, convergence may be slow and depends critically on the choices of step size.

Experimental evidence supporting this characterization of the ascending proxy auction as slow to converge is given by Hoffman et al [**29**], who also suggest methods for accelerating the convergence of the procedure. Given that the proxy information is available to the auctioneer for use in an integer programming formulation, they suggest that several instances of the winner determination problem be solved to determine an efficient allocation and VCG prices. The VCG prices are then used as starting prices for winning bundles in the ascending proxy auction. The bids of losing bidders can be priced at the submitted value, while non-winning bundles of winning bidders can start at an initial price equal to their value (as entered into the proxy) minus the current surplus (i.e. bid minus payment) on that bidder's winning bundle. The ascending proxy auction then proceeds as described by Ausubel and Milgrom [**5**], but now from a better starting point determined by solving at most $\min(N, M) + 1$ winner determination problems.

We suggest an alternative to this *accelerated* proxy auction using a constraint generation technique. Starting at the VCG payments as in [**29**], we wish to achieve a "bidder-Pareto-optimal core outcome." Noting that the core region in payment space can be defined by an exponential number of linear constraints (one for each coalition $C \subseteq J = \{1, ...M\}$), we are motivated to generate the constraints only when they are violated for a given payment vector, rather than applying all constraints to define the core region as in Parkes [**54**]. To do this, we formulate the separation problem for core constraints (problem SEP^t in §5.1), an IP which can be solved to find the most violated constraint. Payments are then adjusted by solving an LP with constraints added from each solution of SEP^t. When no new constraints can be generated, the procedure terminates at a core outcome.

The desirable bidder-Pareto-optimal core outcome obtained by the accelerated proxy method [**29**] is indeed the payment structure we achieve using the technique of this chapter. Though this method is proposed to determine the final payments in the three-stage auction of Chapter 4, we advocate this outcome for any sealed-bid combinatorial auction. Any payment/allocation that is not in the core is susceptible to the complaints of dissatisfied bidders, while a core outcome is *bidder-Pareto-optimal* if there is no other core outcome weakly preferred by every winning bidder. Whenever the VCG outcome is not in the core, the bidder-Pareto-optimal core outcome discourages bid reduction by minimizing total payments, and corresponds exactly to the VCG outcome whenever it is in the core. As our technique is proposed for a general combinatorial auction scenario, we maintain an XOR of flat bids language (with an exclusive bid for any bundle) throughout this chapter, allowing for a direct comparison to the work of Hoffman et al [**29**] and others.

5.1. Core Constraint Generation

As in Hoffman et al [**29**], our general approach is to first solve several winner determination problems for a sealed-bid combinatorial auction, settling on a particular set of winning bidders and VCG payments. Given this fixed outcome Γ_{VCG} we denote the associated coalition of winners as $W = C_{\Gamma_{VCG}}$ and the associated payments π_j^{VCG} for each bidder $j \in W$. Starting from this VCG outcome, we wish to arrive at a bidder-Pareto-optimal core outcome with the same efficient allocation; only payments will differ between the VCG outcome and the final core outcome proposed here.

From a constrained optimization point of view it is difficult to categorize the bidder-Pareto-optimal core payments for two reasons. First, an exponential number of possible coalitions must be considered to define the core region in payment space. Secondly, it is difficult to gauge how much a winning bidder will contribute to a "coalitional value function" without knowledge of her final payment.

Given our definition of the core, it seems convenient to define a core constraint for any blocking outcome Γ as $\sum_{j \in W} \pi_j \geq z_{C_\Gamma}$, where π_j is a payment made by each bidder $j \in W$, and z_C is the *coalitional value* of C. We define this coalitional value z_C as the maximum total payments that a coalition C would be willing to offer the seller in any outcome Γ with $C_\Gamma = C$. This formulation of a core constraint emphasizes that any offer that could be made by a coalition must be matched by the set of winning bidders. With every coalition defining a constraint of this form, we suggest our first characterization of the core:

(CORE1)
$$\sum_{j \in W} \pi_j \geq z_C, \qquad \forall C \subseteq J$$
$$\pi_j^{VCG} \leq \pi_j \leq b_j(S_j), \qquad \forall j \in W$$

Defining the core points in payment space with linear constraints gets us closer to computing the bidder-Pareto-optimal core payments using an LP, and allows us to restate our two difficulties from above:

- Noting the exponential number of constraints in CORE1, how do we separate which constraints must be applied from those that can be ignored without consequence?
- How do we compute the value of z_C for a particular coalition C?

To appreciate the subtlety of the second question, observe that given a candidate payment vector π^t, any bidder $j \in W$ would not want to join a coalition and receive less surplus than she would at her current payment π^t_j. Trying to solve the winner determination problem restricted to the bidders in the coalition C therefore overstates the amount that any winning bidder in C would contribute to the coalitional value z_C, because she will not be compensated her opportunity cost (the amount of surplus she stands to gain at the current vector of payments). The result would be the application of a too restrictive constraint, therefore charging the winning bidders too much. We must instead consider that each winning bidder's opportunity cost limits her contribution to a blocking coalition. We therefore amend our notation to capture the fact that a coalitional value function depends on a payment vector π, and thus change to $z_C(\pi)$.

We may now describe our general iterative approach as follows: at a current vector in payment space, π^t at each discrete iteration t, find the coalition C^t with the highest coalitional value relative to current payments, $z_C(\pi^t)$. If this coalition blocks the currently proposed allocation (the efficient solution with payments π^t), apply the corresponding core constraint and find a new bidder-Pareto-optimal payment π^{t+1}. If this coalition does not block then terminate at the currently proposed outcome (since it satisfies all core constraints). Before demonstrating this technique in full detail, we observe a few basic properties of each winning bidder's "coalitional contribution" relative to current payments, leading us to a more useful LP characterization than that of CORE1.

Let S_j be the bundle awarded to winning bidder j in the efficient solution. Relative to the current payment vector, bidder j would not voluntarily join a coalition that offers him less surplus than $b_j(S_j) - \pi_j^t$, his opportunity cost. If a coalition provides bidder j with bundle $\bar{S}_j$ (possibly the same as S_j, but in general not) then bidder j would contribute at most $b_j(\bar{S}_j) - (b_j(S_j) - \pi_j^t)$ into the coalition value function; if he contributed more, then he would perceive less benefit (surplus) from the hypothetical outcome of the coalition then the one proposed by the auctioneer at iteration t. If bidder j is not in the set W, there is no opportunity cost to recover, and he would offer his entire value for a bundle in an effort to block the set of winning payments. In this case $b_j(S_j = \emptyset) = \pi_j^t = 0$ and his coalitional contribution would be $b_j(\bar{S}_j)$. In general we have the following definition:

Definition: Facing payment of π_j^t for bundle S_j in the efficient allocation, bidder j would be willing to make a *coalitional contribution* of $q_j(C, \pi_j^t) = b_j(\bar{S}_j) - b_j(S_j) + \pi_j^t$ to receive $\bar{S}_j$ as part of the coalition C.

Lemma 5.1. We note the following basic properties of the coalitional contribution $q_j(C, \pi_j^t)$:

(1) If bidder j were to pay an amount π_j^C that is greater than $q_j(C, \pi_j^t)$ to join coalition C and win $\bar{S}_j$, then bidder j would experience less surplus from the coalition than from the auctioneer's proposed outcome S_j with payment π_j^t

(2) $q_j(C, \pi_j^t)$ increases linearly in π_j^t, $\forall j \in W$

(3) $\forall j \in J$, $\bar{S}_j = S_j$ implies that $q_j(C, \pi_j^t) = \pi_j^t$

(4) $q_j(C, \pi_j^t) = b_j(\bar{S}_j)$ and $j \in W \cap C$ imply that $\pi_j^t = b_j(S_j)$

(5) $q_j(C, \pi_j^t) = b_j(\bar{S}_j)$, $\forall j \notin W$

Proof. Property (1) verifies that our definition of the quantity $q_j(C, \pi_j^t)$ truly reflects what we mean by coalitional contribution, and follows from the definition: $\pi_j^C > q_j(C, \pi_j^t) = b_j(\bar{S}_j) - b_j(S_j) + \pi_j^t$ implies that surplus from the coalitional outcome is $b_j(\bar{S}_j) - \pi_j^C < b_j(S_j) - \pi_j^t$, where the right-hand-side of the last inequality is exactly the surplus from the auctioneer's outcome. Properties (2), (3), and (4) follow directly from the definition while Property (5) follows from the standard assumption that $b_j(\emptyset) = 0 \ \forall \ j$, and thus $\pi_j^t = 0 \ \forall j \notin W$. $\square$

Now that we have established the importance of opportunity cost and shown that bidder j's coalitional contribution will normally be less than her value for the bundle given to her by the coalition, we now formulate the *core-constraint separation* *problem* at payment vector π^t. At any point in payment space, π^t, the integer program SEP^t finds the most violated core constraint, or tells us that no blocking coalition can be found.

$$
\begin{aligned}
(\text{SEP}^t) \qquad & z(\pi^t) = \max \sum_{j \in J} \sum_{S \subseteq I} b_j(S) \cdot x_j(S) - \sum_{j \in W} (b_j(S_j) - \pi^t_j) \cdot \gamma_j \\
\text{subject to} \qquad & \sum_{S \supseteq \{i\}} \sum_{j \in J} x_j(S) \leq 1, \qquad \forall i \in I \\
& \sum_{S \subseteq I} x_j(S) \leq 1, \qquad \forall j \in J \backslash W \\
& \sum_{S \subseteq I} x_j(S) \leq \gamma_j, \qquad \forall j \in W \\
& x_j(S) \in \{0,1\}, \qquad \forall S \subseteq I, \forall j \in J \\
& \gamma_j \in \{0,1\}, \qquad \forall j
\end{aligned}
$$

The added terms in the objective of this IP formulation (starting from a general XOR winner determination problem) ensure that any winning bidder will be compensated his opportunity cost if selected. It is easy to verify that any bidder $j \in J$ contributes exactly $q_j(C, \pi^t_j)$, his coalitional contribution.

If the objective $z(\pi^t) > \sum_{j \in W} \pi^t_j$ then the bidders for which $x_j(S) = 1$ form a coalition C^t which blocks the efficient allocation with the current set of payments; i.e. C^t is a set of bidders such that $z(\pi^t) = z_{C^t}(\pi^t)$. If $z(\pi^t) = \sum_{j \in W} \pi^t_j$ then we have achieved an unblocked (i.e., core) outcome; no coalition of bidders would be able to offer an outcome that both they and the auctioneer would prefer. Also note that $z(\pi^t) < \sum_{j \in W} \pi_j^t$ is not a possibility since the feasible allocation of items to the winning set W achieves an objective value of $\sum_{j \in W} \pi_j^t$. As a result, the algorithm developed in this chapter is ascending in terms of total payments.

To see that this technique can be applied to any winner determination scenario, observe the corresponding re-formulation of the core-constraint separation problem for use in Stage III of the Schedule Auction (from Chapter 4):

(SEP-SA)

$$z(\pi^t) = \max \sum_{(i,j,k) \in I \times J \times K_j} b_{ijk} \cdot x_{ijk} + \sum_{l=1 \text{ to } L} B_l \cdot y_l - \sum_{j \in W} (b_j(S_j) - \pi_j^t) \cdot \gamma_j$$

$$\text{subject to } \sum_{i \in I} x_{ijk} + \sum_{l | (j,k) \in K_l} y_l \leq \gamma_j, \ \forall (j,k) \text{ with } j \in J \text{ and } k \in K_j$$

$$\sum_{j \in J} \sum_{k \in K_j} x_{ijk} + \sum_{l | i \in S_l} y_l \leq 1, \ \forall i \in I$$

$$x_{ijk} \in \{0,1\}, \ \forall i,j,k$$

$$y_l \in \{0,1\}, \ \forall l$$

$$\gamma_j \in \{0,1\}, \ \forall j$$

Comparing with the Stage III winner determination problem P_3 from Chapter 4, we see again that the core-constraint separation problem is simply the winner determination problem augmented with opportunity costs for each bidder j that must be deducted from the objective function whenever j belongs to the coalition implied by a feasible solution. In general, this can be implemented for any combinatorial auction with an IP winner determination problem by at worst including constraints of the form $x_j \leq \gamma_j$ for every decision variable x_j signifying the allocation of an item to bidder j. In both SEP^t and SEP-SA, however, our formulations make more efficient use of the γ_j variables by placing them within existing constraints. In any case, one can employ some version of the core-constraint separation problem to determine a maximally violated blocking coalition C^t.

Having found a coalition C^t that blocks the proposed outcome with payment vector π^t, we know that the constraint $\sum_{j \in W} \pi_j \geq z_{C^t}(\pi)$ is violated for the current payment vector. One way to characterize the set of core payments is therefore as the set of points in payment space that satisfy the constraints formulated above as CORE1. The region defined by CORE1, however, is difficult to generate or optimize over because there are an exponential number of constraints in the first set and each has a right-hand-side requiring optimization of an $\mathcal{NP}$-hard winner determination problem to compute. In addition, Property (2) from Lemma 5.1 indicates that price increases on any bidder $j \in W \cap C$ completely cancel with the corresponding increase in coalitional contribution $q_j(C, \pi_j^t)$ on the right-hand-side of the constraint for C in CORE1.

Consequently, our method is to find bidder-Pareto-optimal core payments by generating core constraints of the form $\sum_{j \in W \setminus C^t} \pi_j \geq z(\pi^t) - \sum_{j \in W \cap C^t} \pi_j^t$. The right-hand-side of these constraints are computed once and *remain constant* for the remainder of the algorithm, unlike the right-hand-side of the constraints in CORE1. In economic terms, we choose these constraints because they do not hold winning bidders in a blocking coalition C^t responsible for overcoming coalition C^t, the futility of which is clear by our formulation of coalitional contribution. We also note that these constraints are equivalent to those defined as core constraints in Parkes [**54**] and Hoffman et al [**29**] through a linear change of variables.

After finding each of these constraints using SEP^t, we may solve the following linear program BPO^t to find a set of bidder-Pareto-optimal core payments, in the core relative to all coalitions found through iteration t:

(BPO^t)
$$\theta^t = \min \sum_{j \in W} \pi_j$$

(CORE2) subject to
$$\sum_{j \in W \setminus C^\tau} \pi_j \geq z(\pi^\tau) - \sum_{j \in W \cap C^\tau} \pi_j^\tau, \qquad \forall \tau \leq t$$

$$\pi_j^{VCG} \leq \pi_j \leq b_j(S_j), \qquad \forall j \in W$$

We may then use the value of each π_j in the solution for the next iteration (i.e. set $\pi_j^{t+1} = \pi_j$). We now show that the resulting ascending algorithm converges to the desired outcome.

Theorem 5.2. If $z(\pi^t) = \theta^{t-1}$, then the solution to BPO^{t-1} yields bidder-Pareto-optimal core payments.

Proof. First we note that the payments π^t are in the core, because if not a blocking coalition provides a solution to SEP^t greater than θ^{t-1}. By the minimality of $\sum_{j \in W} \pi_j$ with respect to the constraints (CORE2), if any bidder experiences a payment reduction without an increase in the payment of some other bidder, then some constraint from (CORE2) must be violated. Thus, we find that the payments must be Pareto-efficient with respect to core constraints. The convergence of this algorithm is guaranteed because only a finite number of constraints may be generated, and because the region (CORE2) always contains at least the trivial pay-as-bid solution. □

In some cases, there may be multiple optimal solutions to BPO^t, for which we suggest the following refinement, a linear program EBPO^t which finds *equitable* bidder-Pareto-optimal core payments:

$$(\text{EBPO}^t) \qquad \theta^t(\varepsilon) = \min \sum_{j \in W} \pi_j + \varepsilon m$$

$$\text{subject to} \qquad \sum_{j \in W \setminus C^\tau} \pi_j \geq z(\pi^\tau) - \sum_{j \in W \cap C^\tau} \pi_j^\tau, \qquad \forall \tau \leq t$$

$$\pi_j - m \leq \pi_j^{VCG}, \qquad \forall j \in W$$

$$\pi_j \leq b_j(S_j), \qquad \forall j \in W$$

$$\pi_j \geq \pi_j^{VCG}, \qquad \forall j \in W$$

with decision variables appearing only on the left-hand-side of each constraint. By taking a small enough value of the scalar ε, we have that $\sum_{j \in W} \pi_j$ computed from the solution to this problem is equal to θ^t from the corresponding instance of BPO^t. We then see that the effect of the new terms is to find a set of payments that minimizes the maximum difference from the VCG payments over all bidders, among all outcomes that minimize the total payments of winning bidders. The entire process for determining equitable bidder-Pareto-optimal core payments is summarized in Figure 5.1.

In the next section we illustrate this core constraint generation algorithm with an example, and compare the results to those of the minimax (threshold) rule

Figure 5.1. The Core Constraint Generation Algorithm

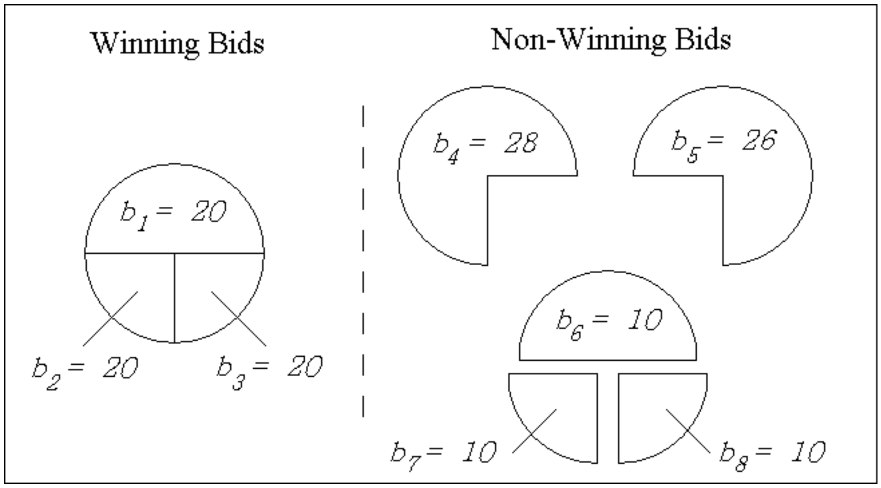
Initialize: Iteratively solve the Winner Determination Problem, finding winners W and VCG payments; Set $t = 1$, $\pi^1 = \pi^{VCG}$, and $\theta^0(\varepsilon) = \sum_{j \in W} \pi_j^{VCG}$
Iteration t: Solve SEP^t If $z(\pi^t) > \theta^{t-1}(\varepsilon)$ Add constraint $\sum_{j \in W \setminus C^t} \pi_j \geq z(\pi^t) - \sum_{j \in W \cap C^t} \pi_j^t$ to EBPO^t and solve, updating $\theta^t(\varepsilon)$. Set $\pi_j^{t+1} = \pi_j$, the values found from EBPO^t. Iterate: $t = t + 1$ Else Terminate

for one-sided auctions as discussed by Parkes [**54**] and to the examples presented in Hoffman et al [**29**].

5.2. Examples and Comparison to the MiniMax Rule

To illustrate our technique for determining bidder-Pareto-optimal core outcomes using core constraint generation, consider the three-item, eight-bidder example of Figure 5.2. Here we depict the three items as pieces of a pie to indicate visually which bids are on which items. Inspection shows that bidders 1, 2, and 3 constitute the set of winners in the efficient allocation, as indicated. In accord with our earlier notation we will say that $W = \{1, 2, 3\}$.

Figure 5.2. An auction for which threshold payments do not minimize total payments



With the set of winners determined, we next compute $\pi_1^{VCG} = \pi_2^{VCG} = \pi_3^{VCG} = 10$, and so set $\pi_1^1 = \pi_2^1 = \pi_3^1 = 10$. Solving SEP^1 the separation problem at iteration 1, we find the most violated blocking coalition $C^1 = \{3, 4\}$ with $z(\pi^1) = 38 > 30 = \sum_{j \in W} \pi_j^1$. We then add the constraint $\pi_1 + \pi_2 \geq 28$ to the formulation EBPO^1 and solve to find the new set of payments $\pi_1^2 = \pi_2^2 = 14$, $\pi_3^2 = 10$. Next we solve SEP^2 and find that the coalition $C^2 = \{2, 5\}$ blocks the current set of payments with $z(\pi^2) = 40 > 38 = \sum_{j \in W} \pi_j^2$. We form EBPO^2 by adding the constraint $\pi_1 + \pi_3 \geq 26$ and solve, yielding $\pi_1^3 = 16$, $\pi_2^3 = 12$, and $\pi_3^3 = 10$. Finally, solving SEP^3 we find that no blocking coalitions exist; the process has terminated at a bidder-Pareto-optimal core outcome.

The algorithm we have just demonstrated selects a specific Pareto-efficient core outcome. Elsewhere in the recent literature, different methods of selecting core outcomes are suggested [**54**], at times resulting in different outcomes than the ones generated by our technique. We compare our method now to the "minimax" rules for finding Pareto-efficient core outcomes that are analogous to the rules developed for finding balanced-budget solutions in combinatorial exchanges [**56**]. For example, reformulated in our own notation, the *threshold rule* selects payments which solve the following optimization problem:

$$\text{(THRESH)} \qquad \min \max_{j \in W} (\pi_j - \pi_j^{VCG})$$
$$\text{subject to} \quad \sum_{j \in W \setminus C} \pi_j \geq n_C, \ \forall C \subseteq J$$

where n_C denotes the portion of the coalitional value for C that is attributable to non-winners. This rule for payment selection can computed as an LP:

$$\text{(THRESH-LP)} \qquad \min m$$
$$\text{subject to} \quad \sum_{j \in W \setminus C} \pi_j \geq n_C, \ \forall C \subseteq J$$
$$\pi_j - m \leq \pi_j^{VCG}, \ \forall j \in W$$

Solving this LP for the above example, we find the solution $\pi_1 = \pi_2 = \pi_3 = 14$, which is unfortunately Pareto-inefficient. There is however a Pareto-efficient point among the optimal solutions to THRESH-LP, and we may next consider the strength of the threshold rule as a selection criteria among the multitude of points on the Pareto-frontier of the core. As in our own formulation for selecting equitable payments, this may be accomplished with the insertion of a secondary objective weighted by a tiny value ε:

(Pareto-THRESH)
$$\min m + \varepsilon \cdot \sum_{j \in W} \pi_j$$
$$\text{subject to} \quad \sum_{j \in W \setminus C} \pi_j \geq n_C, \ \forall C \subseteq J$$
$$\pi_j - m \leq \pi_j^{VCG}, \ \forall j \in W$$

Solving this new LP for the example of this section, we arrive at the solution $\pi_1 = \pi_2 = 14$, and $\pi_3 = 12$ which is indeed Pareto efficient and minimizes the maximum deviation from the VCG payments.

The selection of a Pareto-efficient outcome is by its very nature a matter of taste. Pareto-efficient points are by definition unable to be compared with a strict dominance relationship; movement from one Pareto optimal point to another results in an increase in utility to one bidder if and only if another bidder experiences a decrease in utility. We note, however, the following observation:

Proposition 5.3. A core outcome that minimizes the total payments by bidders will be strictly preferred by the bidders to any other Pareto-efficient core outcome in the presence of side payments.

Compare, for example, the outcome arrived at by the solution of EBPO^t, $\pi_1 = 16$, $\pi_2 = 12$, $\pi_3 = 10$, and the outcome arrived at by the solution of Pareto-THRESH, $\pi_1 = \pi_2 = 14$, $\pi_3 = 12$. In the former case the sum of the payments equals 38 while in the latter case the payments sum to 40. If asked which outcome is preferred, in the presence of side payments the winners will select the former. This is because the bidders that stand to benefit from the change (bidders 2 and 3, in this case) can pay off the bidders who would experience a payment increase (just bidder 1) to make them indifferent to the change. In this case, bidders 2 and 3 could compensate bidder 1 a payment of 1 unit each in an effort to coax her into a move from the latter to the former. Bidder 1 would then be indifferent while bidders 2 and 3 will each still gain 1 unit of surplus from the change. We note that the same comparison holds for the "equal-pay rule" discussed by Parkes [**54**] (in which the maximum difference between the bundle values and payments is minimized), since by the construction of our example all winning bidders have the same valuations and VCG payments.

Having established with the previous example that the outcome of our procedure may deviate from the threshold outcome as formulated in Parkes [**54**] and Hoffman et al [**29**], we now demonstrate that our technique may provide computational advantages over existing iterative methods. Consider the following example from Hoffman et al [**29**]:

Bidder	1	2	3	4
Package	AB	BC*	AC	A*
Value	20	26	24	16

where * in their notation denotes the winners in the efficient allocation. They present a comparison of several methods for obtaining core outcomes, and here we present their results for this problem in terms of number of rounds, along with the VCG and threshold payments, with an added row displaying our own results using core constraint generation. In Figure 5.3 we see that where each of their iterative techniques may require several rounds to solve this problem, Core Constraint Generation terminates after a single round of price adjustments. Starting at the VCG payments, $\pi_2^{VCG} = 8$, $\pi_4^{VCG} = 0$, we find the most violated blocking coalition consisting of just bidder 3. We then equitably divide the burden of overcoming this coalition (using EBPO) and find that no other blocking coalition exists. Notice that this procedure obviates the need to consider a constraint for the coalition $\{1\}$ which is made redundant by the constraint of coalition $\{3\}$.

Comparisons to all the examples worked out by Hoffman et al [**29**] verify this seeming dominance of the Core Constraint Generation technique; for every problem instance presented fully there, Core Constraint Generation terminates after a single

Figure 5.3. Comparison of Core Constraint Generation to proxy methods

Method	Rounds	Revenue	Payment by 2	Payment by 4
Pure Proxy	3100	24.02	12.01	12.01
Safe Start	800	24.02	16.01	8.01
Increment Scaling	20	24.02	17.01	7.01
Increment Scaling w/Safe Start	15	24.02	16.01	8.01
VCG payments	-	8.00	8.00	0.00
Threshold Payments	-	24.00	16.00	8.00
Core Constraint Generation	1	24.00	16.00	8.00

price adjustment. This is in part an artifact of the size of the problems; each iteration of our technique identifies a blocking coalition and only a few coalitions block the VCG outcome in these small problems. Still, we argue that Core Constraint Generation dominates these techniques for small problems and look forward to a more thorough and larger scale comparison in the future. Indeed, we believe that the strength of Core Constraint Generation will become more evident with large scale problems, as the enumeration of all coalitional constraints will become increasingly difficult.

We also note that in no case presented in [**29**] do our computed payments differ from the threshold payments. We propose therefore that there is a phenomenon present in the example of Figure 5.2 that is missing in the examples of [**29**], and note that there may be a problem with the threshold rule as formulated previously for use in this context. In particular, the property of total payment minimization which holds for the threshold rule with respect to budget balance in a combinatorial exchange (as in Parkes, Kalagnanam, and Eso [**56**]) does not hold for the threshold rule with respect to core constraints in a one-sided auction (as formulated in Parkes [**54**] and Hoffman et al [**29**]). Since our EBPO payment rule and the threshold rule share the same intention, it appears that the example of Figure 5.2 illustrates a previously unnoticed property of the threshold rule, and suggest that a refinement of its formulation in which total payments are minimized explicitly is needed in the context of one-sided auctions and core constraints.

5.3. Remarks on Core Constraint Generation

The example of Figure 5.2 helps to justify our proposal that total payment minimization within the core should be the "primary" objective of payment determination with payment equity following as a "secondary" objective, used only to distinguish between multiple optima. Perceived fairness may be an even stronger motivation to adopt such a payment rule. For governmental auctions in particular, many will take comfort that the auctioneer is not selecting an outcome according to self interest, but in the combined best interests of the bidders. In private sector B2B auctions, on the other hand, assurance of an auction that reaps benefits in a minimal fashion may be the best method to maintain long-term trade relationships and encourage repeated participation. Only the core constraints, representing competition, should drive up the payments, not the artificial device used to determine payments.

Additionally, we feel that our method offers computational benefits over other techniques described in the recent literature. By solving the separation problem at each point in payment space we only consider coalitions that *actually* threaten to block a potential outcome, obviating the need to apply the entire exponential set of constraints. We feel that the formulation of the separation problem for violated core constraints and the development of a price adjustment procedure utilizing this approach are novel contributions to the literature. This separation paradigm has proven useful elsewhere for the solution of optimization problems with an exponential number of constraints (e.g. traveling salesman problems), and we hope that forthcoming experiments will verify this for Core Constraint Generation.

Further, our price adjustment procedure takes place without the need for a price increment as in the ascending proxy technique, accelerated or otherwise. EBPO^t upholds every generated core constraint for the remainder of the procedure. This ensures that the same coalition will not appear repeatedly as blocking in our procedure. Experience with the ascending proxy method, on the other hand, shows that after an increment and re-solution of the winner determination problem, the same coalition may appear for several iterations. Instead, our method finds the interesting "change points" implicitly and prescribes adequate payment adjustments at each iteration, rather than requiring the guess of an appropriate increment as a parameter.

As a final note on our comparison to ascending proxy techniques, we do recognize that our method has inferior privacy-preservation properties to the ascending proxy auction as presented by Ausubel and Milgrom [**5**]. Their method can be implemented such that demand is only revealed incrementally to the auctioneer, while our method requires the auctioneer to "open up the proxy" and use all information submitted by the bidders to determine the outcome. We are motivated, however, by Hoffman et al [**29**], who show that the ascending proxy auction can be accelerated by opening up the proxy for use in winner determination. As their research suggests, if it is possible to trust a third party to securely determine the auction outcome without unnecessary public knowledge of the bid submission, then access to all bid information can lead to a more rapid conclusion of a sealed-bid combinatorial auction. We hope that future computational studies will confirm that Core Constraint Generation can accelerate the determination of a bidder-Pareto-optimal core outcome in a sealed-bid auction.

CHAPTER 6

Matrix Bidding

In Chapter 3, we revealed that because of the gross substitutes property bid tables cannot express preferences for complementary bundles, a potential weakness. In Chapter 4, we showed how a more complete set of preferences could be supported in an auction with multiple stages used to reveal the different types of preferences which could not be achieved in a bid table. In this chapter, we explore an alternative method for expressing mixed (both positive and negative) synergy, while still utilizing the compactness provided by a price-vector agent format. We accomplish this by imposing a form of ranked cooperation among the price-vector agents and find that we are rewarded with a wide array of possible preference expression in a compact format that we call Matrix Bidding.

We begin this chapter with the basics of matrix bids and motivate how they can be used with a few examples. In §6.2, we provide an IP formulation of the winner determination problem for a Matrix Bid Auction (MBA) and show that in general this problem is $\mathcal{NP}$-hard. In §6.3 we provide a deeper exploration of the preference expression afforded by the matrix bid format and compare the logical language of matrix bids to other logical languages from the literature.

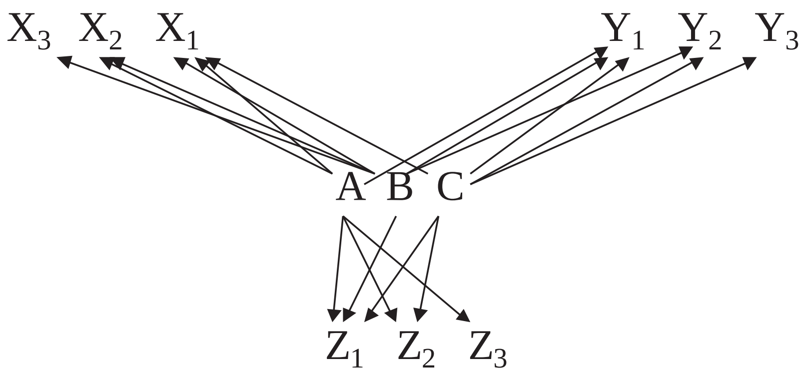


Figure 6.1. Assignment network for a Matrix Bid Auction. Items A, B, and C are to be assigned to agents 1, 2, 3, of bidders X, Y, and Z. The ranked order of items can be inferred from the arcs.

6.1. Interpreting a Matrix Bid

Similar to an approach used in the many-to-one matching literature [**60**] and in our earlier treatment of bid tables, our first step is to represent each bidder by a collection of N agents, each of which can potentially consume a single item. This suggests an assignment network (see Figure 6.1), with N origin nodes (one for each item) and MN destination nodes (one for each of the bidder's agents). As in the context of bid tables, each agent will be represented by a particular column of a matrix bid, and we therefore refer to agents and columns interchangeably.

In contrast to the bid table format, where any permutation of columns would not effect the preferences conveyed by a bidder, here we elicit a bidder specified ordering of the auctioned items and impose a strict ranking among a bidder's agents, determining the order in which the agents may receive items. Specifically, an agent may not receive an item unless the next higher ranked agent receives a higher ranked item.

To define the relationship between items and agents more precisely, we say that each bidder j in this format must submit a numerical ranking r_{ij} for each of the N items at auction. We say that $r_{ij} = 1$ if item i is the first (highest ranked) item in bidder j's ordering of the items, $r_{ij} = 2$ if item i is the second item in bidder j's ordering, etc. We require that this ranking be strict and total for each bidder j: if $i_1 \neq i_2$, then $r_{i_1j} \neq r_{i_2j}$, and $\forall n \leq N$, $\exists i \in I$ with $r_{ij} = n$. It is important to note that the rankings specified by the values of r_{ij} are intended only to guide the coordination of the ranked agents and do not necessarily reflect any preference ordering among the items for that bidder. Stated differently $r_{i_1j} < r_{i_2j} \not\Rightarrow v_j(\{i_1\}) \geq v_j(\{i_2\})$.

For each of bidder j's agents (j,k), define the binary acceptance variable $x_{ijk} = 1$ if agent (j,k) receives item i, and $x_{ijk} = 0$ otherwise. With this terminology defined, we may now state the *cooperation restriction* among a particular bidder's agents:

$$\forall k > 1,\ x_{ijk} = 1 \Rightarrow x_{\bar{\imath}jk-1} = 1 \text{ for some } \bar{\imath} \text{ with } r_{\bar{\imath}j} < r_{ij}$$

This says that any agent (other than agent 1) may not receive an item unless the next more highly ranked agent receives a more highly ranked item. The preference restriction imposed in the matrix bid auction is that the value an item brings to a bidder is determined only by its ranking relative to the other items in the bundle (i.e. which agent it is assigned to). Although this may at first seem restrictive, we will show that when used effectively a great deal can be said by each bidder.

A bidder in this model specifies her preferences with a value for each item in each of its possible rankings in the final bundle. The bid offered for item i by bidder j given that it is the kth best item she receives will be denoted b_{ijk}. Bidder j's bid on a bundle S may then be computed as:

$$b_j(S) = \sum_{i \in S} b_{ijk(i,S)}$$

where $k(i, S)$ gives the ordinal ranking of item i among the items in S. For example, $k(\bar{\imath}, S) = 1$ if no item in S has a higher rank (smaller value of r_{ij}) than item $\bar{\imath}$. Similarly, $k(\bar{\imath}, S) = 2$ if exactly one item has a higher rank than item $\bar{\imath}$, etc.

Each bidder submits an ordered list of the items to establish the values of r_{ij} and a matrix containing non-negative values of b_{ijk} (without loss of generality we assume integer values throughout). The matrix of b_{ijk} entries together with the precedence ordering r_{ij} is referred to as a *matrix bid*. In each matrix bid a row will be associated with a particular item, and each column will be associated with an agent. The item associated with a given row is written to the left of that row, with the row of the highest ranked item on top, the second highest item beneath it etc. Since the kth item in the ordering can be at most the kth item in any bundle, the matrix of b_{ijk} values will be lower triangular.

The following simple rules summarize how to interpret a matrix bid:

- When an item is awarded to a bidder, the auctioneer receives a single bid from the corresponding row in that bidder's matrix bid.
- Only a single bid may be taken from any column
- Except for bid entries in the first column, a bid may not be used unless a bid in the previous column and a higher row is also used.

Example: Suppose a bidder is submitting preferences for the following entertainment choices on a specific date: a ticket to the afternoon baseball game, a coupon for dinner at a nearby restaurant, a day-pass to a water-park (outside of town), and a ticket to a matinee at the local theatre. The bidder reasons that the matinee and baseball game conflict; she cannot go to both, but can make it to dinner after either one. She decides that if she gets any of the other items she will not leave town to go to the water-park. Her matrix bid may appear as follows:

baseball	40			
matinee	10	0		
dinner	25	25	25	
water-park	30	0	0	0

The order r_{ij} is given in the outside column with baseball being ranked first, the matinee second and so forth. The first column inside the matrix (always) gives the bid on the item in that row if it is the highest ranked (or only) item received. The second column gives the price for each item in the row given that it is the second highest item received etc.

If she receives a baseball ticket she is willing to pay 0 for the matinee which she cannot attend due to conflict. If she receives either baseball or matinee in the first column she would being willing to pay 25 for the meal (in the second column). Although the seller is unlikely to give away the matinee ticket, the mathematical formulation does not necessarily rule this out. If the auctioneer gives her the baseball ticket at 40 and the matinee ticket at 0, she is still willing to pay 25 for the dinner, and expresses this with a 25 in the third column; a free matinee ticket does not change her preferences for the dinner. The fourth row shows that she would pay 30 for the water-park pass by itself, but would pay 0 for it if any other items are won. The bid of 30 cannot be accepted with any other bid by the rules outlined above.

Example: A major television network is hosting a worldwide television event with a limited number of available advertising time-slots; for simplicity we assume there are four time-slots, A, B, C, and D, chronologically. They attract companies X, Y, and Z who have very different marketing strategies determining their preferences for these time-slots.

$BidderX$ has developed a two-part "gag" commercial which they will only use if they can secure exactly two time-slots. They are indifferent to which two.

BidderY feels that slots B and C, occurring in the middle of the program are more effective than A and D at the beginning and end of the program when less viewers are watching. The effectiveness of their ads diminishes with repeat viewing so they are uninterested in purchasing more than two slots.

BidderZ agrees that slots B and C are superior to A and D. Their hypnotic ad campaign will show increased effectiveness with each viewing, as the increasing values in each row of their matrix bid submission suggest.

The matrix bids these companies submit are:

Bidder X				
A	0			
B	0	30		
C	0	30	0	
D	0	30	0	0

Bidder Y				
B	20			
C	20	6		
D	10	4	0	
A	10	4	0	0

Bidder Z				
B	7			
C	6	7		
A	5	6	7	
D	4	5	6	7

These examples show the ability of the matrix bid format to express several types of preferences. The first example shows the ability to model a precedence relation where certain items preclude one another while others do not. In the second example the items are thought of as substitutes by Bidder Y, complements by Bidder Z, and somewhere in between by Bidder X who shows preference for a specific quantity.

In order to determine the winners of an MBA one must solve the winner determination problem. By inspection we find that the optimal value for the winner-determination problem of the television advertisement MBA is 57. Bidder X receives items A and D, contributing 30 to the objective function; Bidder Y receives item C for 20 units, while Bidder Z pays 7 units for B. In the next section we present an integer programming formulation for finding the winners in an MBA.

6.2. Integer Programming Formulation of Winner Determination

The winner-determination problem, in this context, is to find the maximum revenue assignment of items to agents, satisfying supply of one for each item and demand of at most one for each agent, as well as upholding the ordering or ranking of items. According to the ranking, a feasible assignment may assign an item to a given agent only if the agents for a particular bidder with a lower value of k are assigned items with a lower value of r_{ij}. These constraints are formalized in the following IP which we will refer to as MBA:

$$\text{(MBA-IP)} \qquad \max \sum_{i \in I} \sum_{j \in J} \sum_{k \leq r_{ij}} b_{ijk} \cdot x_{ijk}$$

$$\text{(6.1)} \qquad \text{subject to} \quad \sum_{j \in J} \sum_{k \leq r_{ij}} x_{ijk} = 1 \text{ for each } i \in I$$

$$\text{(6.2)} \qquad \sum_{i \in I | r_{ij} \geq k} x_{ijk} + y_{jk} = 1 \text{ for each } (j,k) \in J \times K$$

$$\text{(6.3)} \qquad \sum_{l \in I | k \leq r_{lj} \leq r_{ij}} x_{ljk} - \sum_{l \in I | k-1 \leq r_{lj} < r_{ij}} x_{ljk-1} \leq 0 \text{ for each } (i,j,k) \text{ with } k > 1$$

$$x_{ijk}, y_{jk} \in \{0,1\}, \quad \forall i,j,k$$

As above, x_{ijk} denotes binary acceptance variable; $x_{ijk} = 1$ if the bid b_{ijk} is accepted, while $x_{ijk} = 0$ if it is not. Constraint (6.1) is a set of supply constraints for each item, while demand constraints (6.2) assure that at most one item is received by each agent. In each demand constraint (6.2) we introduce a binary slack variable y_{jk} which equals one exactly when no item is received by bidder j's kth agent, and is zero otherwise.

Constraints (6.3) enforce the ordering; $x_{ijk} = 0$ unless some $x_{lj(k-1)} = 1$ where $r_{lj} < r_{ij}$. This is not the only way to formulate this set of constraints and not the most obvious; these inequalities are chosen to eliminate as many fractional solutions as possible while enforcing the ordering by including more than is necessary in the first summation. Padberg and Alevras [**52**] study the "order preserving assignment problem" which corresponds exactly to an instance of MBA-IP with only a single bidder, and potentially less than N agents. They show for their own formulation that the ordering may be enforced with constraints like those of (6.3), and in particular that constraints in this form are facet defining. To see that this formulation of the ordering constraints is superior to the "natural" version (i.e. $x_{ijk} - \sum_{l \in I | k-1 \leq r_{lj} < r_{ij}} x_{ljk-1} \leq 0$ for each (i, j, k) with $k > 1$) observe that in a three item auction for items A, B, and C with $r_{A1} = 1$, $r_{B1} = 2$, and $r_{C1} = 3$ it is possible to have a fractional solution with $x_{A11} = x_{B12} = x_{C12} = \frac{1}{2}$ using the "natural" constraints while the constraints (6.3) do not admit such a solution.

Although the assignment problem is polynomially solvable, adding constraints (6.3) greatly complicates the matter. In particular, it is not hard to find examples in which the LP-relaxation of MBA-IP will produce a fractional optimal solution. Interestingly, Padberg and Alevras [**52**] show that the order preserving assignment problem is polynomially solvable (Lemma 6.2 below generalizes their result), while Theorem 6.1 shows that the winner determination problem for an MBA is $\mathcal{NP}$-hard. In other words evaluating one's own matrix bid can be done in polynomial time, while finding an assignment that simultaneously satisfies several orderings (one for each matrix bid) represents a hard decision problem for the auctioneer.

We now verify one strength of the formulation MBA-IP: it is polynomially-sized in the number of items and bidders, unlike the general winner determination problem (GWD) which requires an exponential number of variables, one for every bidder/bundle pair. This suggests that for some values of N a GWD auction cannot be performed due to memory restrictions, while an MBA may still be held for the same set of N items. To verify this, we note that each bidder must submit each matrix bid entry or $1+2+3+...N$ different bids due to the triangular shape of each bid matrix. Each bidder thus submits $\sum_{l=1}^{N} l = \frac{N(N+1)}{2}$ bid entries and an ordered list of N items, $\frac{1}{2}N^2 + \frac{3}{2}N$ pieces of information. The total number of x variables is equal to the total number of matrix bid entries $M \cdot \frac{N(N+1)}{2}$, while the total number of y variables is simply MN. Counting the constraints similarly according to their indices we find that there are N of type (6.1), MN of type (6.2), and $M \cdot \frac{N(N-1)}{2}$ of type (6.3). Thus, MBA-IP has $\frac{1}{2}N^2M + \frac{3}{2}NM$ variables and $\frac{1}{2}N^2M + \frac{1}{2}NM + N$ constraints. Our computational results (Chapter 7) suggest that for reasonable N and M, MBA-IP instances of this size can be solved quickly enough for auction implementation purposes, despite the following theorem.

Theorem 6.1. The winner determination problem for a matrix bid auction is $\mathcal{NP}$-hard.

Proof. We employ the $\mathcal{NP}$-hard Weighted Set-Packing Problem (WSP): Given a set S, and a list of subsets of S: S_1, S_2, ... S_p, each with a corresponding weight w_1, w_2,

... w_p, problem WSP asks for the maximum weight collection of mutually-disjoint S_is.

The transformation of this general instance of WSP into a matrix bid auction proceeds in a straightforward manner. Identify S from the WSP with I and create p bidders, one for each S_i. Each (single-minded) bidder will submit a matrix bid with the elements of this particular S_i shuffled to the top of the matrix ordering, and a bid of w_p placed in the $|S_i|$th diagonal entry. An optimal solution to the winner-determination problem for this auction MBA-IP provides a maximum weight set-packing of S_1, S_2, ... S_p. $\square$

Despite this classification of the decision problem as $\mathcal{NP}$-hard, our experience tells us that the formulation MBA-IP is fairly well-behaved. It is well known that the assignment constraints (6.1) and (6.2) on their own define a feasible region with integer extreme points. With the addition of constraints (6.3) many of these extreme points are removed and several fractional extreme points are introduced to the polyhedron. Exactly how many of each type are generated or eliminated by (6.3) varies among instances (even of the same size) depending on the orderings chosen by bidders. Every integer solution to MBA-IP, however, is also a feasible assignment and thus an extreme point of the region defined without (6.3), partially explaining the proliferation of LP-relaxation solutions that are integer optimal as seen in our experiments. Another partial explanation is suggested by the following lemma.

Lemma 6.2. For any bidder j, if $\sum_k x_{ijk} \in \{0,1\}$ for all items $i \in I$, then $x_{ijk} \in \{0,1\}$ for all i and k.

Proof. The implication is trivial for all i such that $\sum_k x_{ijk} = 0$, so we consider the case in which $\sum_k x_{ijk} = 1$.

Let $i_1 = \arg\min_{I_j} r_{ij}$, where $I_j = \{i | \sum_k x_{ijk} = 1\}$ (i.e. I_j is the set of all items awarded to j). Similarly, let $i_2 = \arg\min_{I_j \setminus \{i_1\}} r_{ij}$, $i_3 = \arg\min_{I_j \setminus \{i_1, i_2\}}$ and so forth, yielding in general that i_n is the nth ranked item in I_j.

First note that since there is no item l with $r_{lj} < r_{i_1 j}$ and $x_{ljk} \neq 0$, constraints (6.3) for i_1 imply that $x_{i_1 jk} \leq 0$ for all $k > 1$, and therefore $x_{i_1 jk} = 0$ for all $k > 1$. Since $\sum_k x_{i_1 jk} = 1$, it must be the case that $x_{i_1 j1} = 1$. By (6.2) it also follows that $x_{ij1} = 0$ for all $i \neq i_1$.

Next, we claim that if $x_{i_n jn} = 1$ for all $n < \overline{k}$, then $x_{i_{\overline{k}} j \overline{k}} = 1$. This is easily shown since $x_{i_n jn} = 1$ for all $n < \overline{k}$ implies that:

$$x_{ijk} = 0 \text{ for all } k \geq \overline{k} \text{ and } i \text{ with } r_{ij} < r_{i_{\overline{k}} j} \tag{6.4}$$

$$x_{i_{\overline{k}} jk} = 0 \text{ for all } k < \overline{k} \tag{6.5}$$

The first of these two facts (6.4), together with (6.3), implies that $x_{i_{\overline{k}} jk} = 0$ for all $k > \overline{k}$. Using this with (6.5) and $\sum_k x_{i_{\overline{k}} jk} = 1$ we obtain the desired result, $x_{i_{\overline{k}} j \overline{k}} = 1$. The lemma now follows by induction. □

Lemma 6.2 states roughly that if a bidder j receives integer amounts of all items (among all her agents), the constraints of MBA-IP dictate that the assignment of items to bidder j's collection of agents must be integral. Alternatively, items may not be split fractionally among the agents of a single bidder j, unless there is some item which j shares fractionally with a different bidder.

The polynomial-solvability of the order-preserving assignment, first proved by Padberg and Alevras [**52**], follows as a special case of Lemma 6.2, which implies that the polyhedron defined by constraints (6.1), (6.2), and (6.3) with just a single bidder has integer extreme points. Further, Lemma 6.2 offers an alternate and more succinct proof than Padberg and Alevras [**52**].

As stated here, Lemma 6.2 suggests the possibility of an advantageous branching strategy when using a branch-and-bound technique to solve MBA-IP. Rather than branching on a single variable at each node in the branch-and-bound tree, it seems possible to benefit by branching on a bidder/item pair, dictating that the specific item definitely be awarded to bidder j on one branch, and that the item definitely not be awarded to bidder j on the other branch. It is advantageous to focus on bidder/item pairs corresponding to items split fractionally among several bidders, since by Lemma 6.2, if these bidder/item pairs don't exist, then items will not be split fractionally among the agents of the same bidder, and the resulting solution must therefore be integer.

How to choose such a bidder/item pair to branch on when many candidates are available is itself a difficult question. Indeed, so far in our computational experiments, a robust heuristic for this MBA subproblem has remained elusive. Several versions of a specialized branching strategy have been investigated as part of our research, though none is able to challenge CPLEX 8.0's default branching algorithm (which itself applies several heuristics for faster branching). We believe that the pursuit of a branching strategy based on this observation remains worthy of future study.

6.3. Preference Expression using Matrix Bids

In this section we explore how matrix bids may be used to convey preferences. We suggest the use of matrix bids as "atomic bids" to be combined with the logical connectives, offering a new notational system for writing down preferences that are difficult to express with flat bids alone. We show that a language with matrix bids as atoms is more expressive in a polynomial amount of input than a language with just flat bids as atoms, and equally as expressive as a flat-bid language augmented with "k-of" or "exactly-k-of" bids.

Relative to other logical bidding languages we show that more can be said with a single atom, and that some things can be said easily in matrix bids that are cumbersome in other languages. On the whole we hope to convey that with some experience matrix bids may be an easier to read format than the equivalent collection of bids needed express the same preferences in a language with "k-of" and "exactly-k-of" operators applied to flat bids only. Matrix bids provide a structured way to string together several "k-of", "exactly-k-of", and flat bid sentence fragments to be read in a meaningful way after deleting many of the logical connectives. In §7.2, we will see that this allows us to quickly generate simulated auction data (with no logical connectives) which would have required exponentially long bid sentences in any of the flat-bid auction simulations commonly found in recent literature (e.g. [**28**], [**41**] ,[**65**]).

As first discussed in Chapter 2, let a *flat bid* be denoted (S, p), for an all-or-nothing bid of p monetary units on the set of items S. As noted by Nisan [**50**], certain natural expressions of preference can require exponentially long bid sentences in a logical language with only flat bids as atoms. The work of Boutillier and Hoos ([**9**], [**10**], [**30**]) explore the possibility of expanding the bid language to include a "k-of" operator, handling one of the most natural expressions that are tiresome to convey in flat bids by simply adding it in explicitly. At first, this does not seem that rewarding in terms of the impact on winner determination, because the authors simply expand a k-of expression back into flat bids prior to winner determination. But Boutillier [**9**] later shows how to explicitly model the k-of operator in an IP formulation of winner determination with a new class of constraints to handle the new operator.

In this same spirit we ask: are their other natural preference types that are difficult to handle in a logical language of flat bids? Should the list of connectives be augmented to accept other forms of preferences like the k-of operator? One example of another natural connective is the "exactly k-of" operator, where the k-of operator discussed elsewhere could be called more precisely the "at least k-of" operator (though we will continue to refer to this one as the k-of operator). It is not difficult to show that the exactly k-of operator can be handled in an IP winner determination formulation with specialized constraints as the k-of operator is handled in [**9**].

Adding more and more basic connectives and new classes of constraints seem to complicate the matter. Reading and writing bids in such a language becomes difficult (for a human user) with many connectives and complicated nestings to consider when putting together an entire bid sentence or a complete profile of preferences. Is there instead an atomic format which can accept a wide variety of preferences including in particular those that are difficult to express in flat-bids alone? We propose that matrix bids offer such a format and that a logical language of matrix bids can be made at least as expressive in a polynomial amount of input space as any of the other languages in the literature. This latter statement is justified by comparison to the language $\mathcal{L}_{GB}$ of [**9**], which in turn is at least as expressive as any language that precedes it. In particular, if we let $\mathcal{L}^{flat}$ be the language $\mathcal{L}_{GB}$ of [**9**] without the k-of operator, and $\mathcal{L}_{MB}$ be the language formed with matrix bids as atoms and the same logical connectives as in $\mathcal{L}_{GB}$, we will show that the preferences which can be expressed compactly in $\mathcal{L}^{flat}$ are properly contained in $\mathcal{L}_{MB}$, while the preferences which can be expressed compactly in $\mathcal{L}_{MB}$ are exactly equal to the preferences which can be expressed compactly in $\mathcal{L}_{GB}$.

6.3.1. Capacity Constraints and Capacity Costs

We begin with an illustrative demonstration of one of the strengths of matrix bids relative to the language $\mathcal{L}^{flat}$, which is composed of flat bids joined by the logical connectives AND, OR, and XOR. We show how simple capacity-constraints are naturally expressed in a matrix bid, and compare with the equivalent formulation in $\mathcal{L}^{flat}$.

Suppose we have an auction with $N = 6$ and a bidder wants to express an additive valuation over the items with a constraint that he cannot consume more than 3 items. A matrix bid expressing this (with arbitrary values given for each item) is as follows:

$$\begin{array}{c|cccccc|} A & 22 & & & & & \\ B & 18 & 18 & & & & \\ C & 17 & 17 & 17 & & & \\ D & 16 & 16 & 16 & 0 & & \\ E & 14 & 14 & 14 & 0 & 0 & \\ F & 12 & 12 & 12 & 0 & 0 & 0 \end{array}$$

With this bid, he pays the same cost for each item regardless of what other items he gets, but will never pay a positive amount for a fourth, fifth, or sixth item, reflected by the zeros in the fourth, fifth, and sixth columns. For a comparison of size and readability, we now introduce some helpful notation from [**9**]. Let $A : 22$ denote, for example, a bid of 22 on item or expression A. We may use A by itself in place of $A : 0$. Additionally, let the symbols $\wedge$, $\vee$, and $\oplus$, denote AND, OR, and XOR respectively. These symbols are used to join together subclauses to form sentences in the language $\mathcal{L}^{flat}$, where each subclause is either a singleton bid like $A : 22$, or, recursively, any other sentence in the language. Bid amounts can be placed at any subclause that are accepted whenever the subclause is a valid statement of sentential logic, with truth values of atomic singleton bids assigned to be true if and only if the corresponding item is received. The previous matrix bid expressed in the language and notation of Boutillier [**9**] is now:

$$
\begin{aligned}
&(A:22 \vee B:18 \vee C:17) \oplus (A:22 \vee B:18 \vee D:16) \oplus (A:22 \vee B:18 \vee E:14) \oplus \\
&(A:22 \vee B:18 \vee F:12) \oplus (A:22 \vee C:17 \vee D:16) \oplus (A:22 \vee C:17 \vee E:14) \oplus \\
&(A:22 \vee C:17 \vee F:12) \oplus (A:22 \vee D:16 \vee E:14) \oplus (A:22 \vee D:16 \vee F:12) \oplus \\
&(A:22 \vee E:14 \vee F:12) \oplus (B:18 \vee C:17 \vee D:16) \oplus (B:18 \vee C:17 \vee E:14) \oplus \\
&(B:18 \vee C:17 \vee F:12) \oplus (B:18 \vee D:16 \vee E:14) \oplus (B:18 \vee D:16 \vee F:12) \oplus \\
&(B:18 \vee E:14 \vee F:12) \oplus (C:17 \vee D:16 \vee E:14) \oplus (C:17 \vee D:16 \vee F:12) \oplus \\
&(C:17 \vee E:14 \vee F:12) \oplus (D:16 \vee E:14 \vee F:12)
\end{aligned}
$$

If this example does not yet convince the reader of exponential growth for this type of preference expression in $\mathcal{L}^{flat}$, note that if we were to add another item to the auction and maintain the capacity constraint of three items, we would have to add 15 new XORed clauses to the above statement of preferences, while only adding

7 new numbers (one new row) to the matrix bid. In general, additive preferences for n items with a capacity constraint of k takes $\binom{n}{k}$ clauses, each containing k atomic bids in the language of $\mathcal{L}^{flat}$, or a single matrix bid of size $\binom{n}{2} = \frac{n(n-1)}{2}$, verifying that $\mathcal{L}_{MB}$ can contain preferences in a single matrix bid that require a sentence of exponential length in $\mathcal{L}^{flat}$.

This example demonstrates how to apply a capacity constraint using a matrix bid and is not limited to a situation where the underlying preferences are additive. Though we used an additive structure in this example for simplicity, one could start within any matrix bid expression and "zero out" columns that exceed capacity. Further, matrix bids easily handle a capacity cost structure in place of a hard capacity constraint. Imagine a firm that experiences an underlying additive structure (again for simplicity) for items received (for example, in terms of potential revenue generated), but then wishes to capture storage and maintenance costs based on the number of items in their possession.

Suppose, for example, that in a 6 item auction they experience storage and maintenance costs of 0 for the first two items, but have to employ a new worker (at a cost of 3 units) to maintain every two items they purchase after that, and further must rent a storage area at 1 unit each for the fifth and sixth items. If their additive revenue structure is captured in the following matrix bid:

A	22					
B	18	18				
C	17	17	17			
D	16	16	16	16		
E	14	14	14	14	14	
F	12	12	12	12	12	12

then they can subtract the entries of the following row vector from each entry in the corresponding column to capture the storage and maintenance costs:

$$0 \quad 0 \quad 3 \quad 0 \quad 4 \quad 1$$

Thus if revenue accrues additively with the values from the above matrix bid, and costs are as described above, the preferences of this firm are captured in the following matrix bid:

A	22					
B	18	18				
C	17	17	14			
D	16	16	13	16		
E	14	14	11	14	10	
F	12	12	9	12	8	11

Now, they have compactly submitted a bid with positive value for each of the $2^N - 1$ possible non-empty bundles, while $\mathcal{L}^{flat}$ cannot express the generalized version of these preferences compactly. To be certain, note that the information conveyed in the first three columns requires as much space in a $\mathcal{L}^{flat}$ sentence as in the previous example, and that both examples easily generalize due to the underlying additive preference structure.

Note also that with this example the bidder has conveyed a positive incremental value for any item when added to any bundle that does not contain it. This allows for a more aggressive bidding strategy than is possible (compactly) in the most general language, XOR-of-flat-bids, where the statement "I will pay an incremental amount p if item a is added to any of my previously submitted flat-bids" potentially requires the bidder to double the number of submitted flat-bids. To compare with an OR-of-flat-bids language, note that ORing in a new flat bid (a, p) does not allow for any price-discrimination; the auctioneer may force the bidder to accept the bid of p for item a in conjunction with any other bid offers. Matrix bidding, on the other hand, allows for a different incremental value for an item if taken with a certain set of items, for example, or at a certain capacity level.

6.3.2. Characterization of $\mathcal{L}_{MB}$

Having established that $\mathcal{L}_{MB}$ affords expression not available in a polynomially sentence in $\mathcal{L}^{flat}$, we proceed to show that a language with matrix bids as atoms, $\mathcal{L}_{MB}$, is equally as expressive as the language $\mathcal{L}_{GB}$ of Boutillier [**9**] in an amount of input that is polynomially sized in the number of items. To do this, we must show that an arbitrary $\mathcal{L}_{GB}$ bid sentence can be expressed compactly in $\mathcal{L}_{MB}$, and that an arbitrary $\mathcal{L}_{MB}$ bid sentence may be expressed compactly in $\mathcal{L}_{GB}$. The first statement follows logically from the proof of Theorem 6.1, which shows that an arbitrary flat-bid can be held compactly in a matrix bid. We admit however that this doesn't make good use of the space in a matrix bid. We do, however, present techniques for injecting $\mathcal{L}_{GB}$ preferences into matrix bids more efficiently in §6.3.3. For the reverse statement, we need only show that an arbitrary matrix bid can be expressed in $\mathcal{L}_{GB}$. We accomplish this first in the language $\mathcal{L}_{GB}^{+}$ which we form by adding the "exactly k-of" operator to $\mathcal{L}_{GB}$. We find the resulting translation of a matrix bid into $\mathcal{L}_{GB}^{+}$ to be slightly more readable than the equivalent expression given in $\mathcal{L}_{GB}$.

To write these expressions more succinctly, let $\langle 2, \{A : 2, B : 4, C : 8\}, 3\rangle$ denote, for example, the k-of bid of 3 monetary units if at least 2 of the bids $A : 2$, $B : 4$, or $C : 8$ are satisfied, with A, B, C receiving a bid of 17. Similarly, let $[2, \{A : 2, B : 4, C : 8\}, 3]$ denote the exactly-k-of bid of 3 monetary units if exactly 2 of the subclauses $A : 2$, $B : 4$, or $C : 8$ are satisfied; in this case, the set A, B, C receives a bid of 14.

Given a general matrix bid from a four-item auction:

$$\begin{array}{c|cccc|} A & a_1 & & & \\ B & b_1 & b_2 & & \\ C & c_1 & c_2 & c_3 & \\ D & d_1 & d_2 & d_3 & d_4 \end{array}$$

an equivalent bid in the language $\mathcal{L}_{GB}^{+}$ is given by:

$$\begin{array}{l} (A : a_1) \\ \vee \\ (B \wedge [0, \{A\}, 0] : b_1) \oplus (B \wedge [1, \{A\}, 0] : b_2) \\ \vee \\ (C \wedge [0, \{A, B\}, 0] : c_1) \oplus (C \wedge [1, \{A, B\}, 0] : c_2) \oplus (C \wedge [2, \{A, B\}, 0] : c_3) \\ \vee \\ (D \wedge [0, \{A, B, C\}, 0] : d_1) \oplus (D \wedge [1, \{A, B, C\}, 0] : d_2) \oplus \\ (D \wedge [2, \{A, B, C\}, 0] : d_3) \oplus (D \wedge [3, \{A, B, C\}, 0] : d_4) \end{array}$$

where we remind the reader that, for example, B on its own is equivalent to $B : 0$. We note that a few minor truncations are possible in this expression (for example, $(D \wedge [3, \{A, B, C\}, 0]) : d_4$ is equivalent to $A \wedge B \wedge C \wedge D : d_1$) but that no major truncations are possible (since each entry is used exactly once). We therefore leave it in this form which accentuates the one-to-one correspondence between a general matrix bid entry b_{ijk} and clauses of the form $(i \wedge [k-1, S_i, 0] : b_{ijk})$, where $S_i = \{\bar{\imath} \in I | r_{\bar{\imath}j} < r_{ij}\}$, allowing us to claim inductively that any matrix bid can be expressed polynomially within $\mathcal{L}_{GB}^{+}$. To show the equivalence in polynomially expressability between $\mathcal{L}_{GB}$ and $\mathcal{L}_{MB}$, we simply note that $[k, S, p]$ is equivalent to $\langle k, S, p \rangle \wedge \langle k+1, S, * \rangle$, where we use $*$ to signify a sufficiently large negative value, making $\mathcal{L}_{GB}^{+}$ and $\mathcal{L}_{GB}$ equivalent up to a polynomial transformation. (The use of $*$ will be discussed below; it may be interpreted as a negative number whose absolute value is larger than the sum of all positive bids in the auction.)

We note finally that both $\mathcal{L}_{GB}$ and $\mathcal{L}_{MB}$ make use of the k-of operator applied to arbitrary subclauses. If we compare instead to the language studied earlier by Hoos and Boutillier [**30**] which makes use of the k-of operator applied only to singleton bids (not arbitrary subclauses), we see that this language is equivalent in terms of polynomial expressability to the logical language of matrix bids connected by only ANDs, ORs, and XORs (since the k-of operator applied only to singleton bids can be expressed in a single matrix bid).

6.3.3. Using the Matrix Bid Format Efficiently

The characterization of $\mathcal{L}_{MB}$ as equivalent to $\mathcal{L}_{GB}$ in terms of polynomial expressability suggests that matrix bids may be thought of as a notational compactification of preferences that could before only be expressed with several k-of expressions. In particular, our capacity constraint example in §6.3.1 showed that a matrix bid readily houses several k-of expressions, while simultaneously allowing for price differentiation among particular items to be expressed in the rows of each item. As noted above, however, our claim that any $\mathcal{L}_{GB}$ sentence could be expressed in a polynomial size in $\mathcal{L}_{MB}$ made poor use of the space afforded to us by a matrix bid. We now show a few techniques for injecting several general forms of preferences into a single matrix bid, demonstrating more efficient use of the freedom allowed by the matrix bid format. We begin by showing that an arbitrary AND, OR, or XOR clause of singleton bids can be contained in a single matrix bid, as can an arbitrary k-of-singletons bid or flat bid.

Recall that in the matrix bid language, a flat bid (S, p) can be written:

$$\begin{array}{c|cccccc|} i_1 & 0 & & & & & \\ i_2 & 0 & 0 & & & & \\ \vdots & \vdots & & \ddots & & & \\ i_{|S|} & 0 & 0 & \cdots & p & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array}$$

where $i_1, i_2, ...i_{|S|}$ are the items in S written in any order, and the ellipses in this example represent regions containing only zeros. This expression represents a very inflexible offer, paid only if the exact package S is received.

A more flexible offer is demonstrated by the arbitrary OR bid, $(i_1 : p_1 \vee i_2 : p_2 \vee ... \vee i_n : p_n) : b$. Maintaining consistency with Boutillier [**9**], this bid offers p_l for item i_l taken in any combination with other items, and offers a bonus b if any one of the items is awarded (i.e. if any of the singleton subclauses is satisfied). This OR bid is captured in the matrix bid:

$$\begin{array}{c|cccccc|} i_1 & p_1 + b & & & & & \\ i_2 & p_2 + b & p_2 & & & & \\ \vdots & \vdots & & \ddots & & & \\ i_n & p_n + b & p_n & \cdots & p_n & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array}$$

Here the items $i_1, i_2, ...i_n$ in the OR statement are placed above (in higher ranked rows) than items not in the statement, while again the ordering among the items in the statement is inconsequential. We form this matrix bid by starting with an additive valuation (a p_l placed in every entry of the row for each i_l), and add the bonus amount b to the first entry in each of these rows. This ensures that the bonus must be awarded if at least one item in $i_1, i_2, ...i_n$ is received, and is only awarded once. The rows below i_n contain the items $I \backslash \{i_1, i_2, ...i_n\}$ and are filled with zeros.

Modeling the arbitrary AND of singleton bids combines the previous two approaches. The $\mathcal{L}_{GB}$ sentence $(i_1 : p_1 \wedge i_2 : p_2 \wedge ... \wedge i_n : p_n) : b$ offers an additive valuation over single items $i_1, i_2, ...i_n$, with a bonus amount b offered only if every item in $\{i_1, i_2, ...i_n\}$ is received. We therefore begin with constant rows to express the additive valuation, and add the bonus amount b to the last element on the diagonal:

$$\begin{array}{c|cccccc|} i_1 & p_1 & & & & & \\ i_2 & p_2 & p_2 & & & & \\ \vdots & \vdots & & \ddots & & & \\ i_n & p_n & p_n & \cdots & p_n + b & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array}$$

where again the order of the items is inconsequential, as long as all the items not belonging to the AND statement are placed in zero rows below those that are in the statement.

The arbitrary XOR of n singleton bids $(i_1 : p_1 \oplus i_2 : p_2 \oplus ... \oplus i_n : p_n)$ makes an offer of p_l on each singleton set $\{i_l\}$, but places no bid on any bundle containing more than one item. This is captured in the following matrix bid, in which we only place positive amounts (specifically the bid p_l for the item in that row) in the first column:

$$\begin{array}{c|cccccc|} i_1 & p_1 & & & & & \\ i_2 & p_2 & 0 & & & & \\ \vdots & \vdots & & \ddots & & & \\ i_n & p_n & 0 & \cdots & 0 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array}$$

where as always, items not involved in the bid are placed at the bottom of the matrix bid with zero rows.

To model the k-of preference $\langle k, S, p\rangle$, a bid of p on at least k items from the set S, consider the matrix bid:

$$
\begin{array}{c|cccccc|}
i_1 & 0 & & k\text{th} & \text{col} & & \\
i_2 & 0 & \ddots & \downarrow & & & \\
\vdots & \vdots & & p & & & \\
\vdots & \vdots & & p & \ddots & & \\
\vdots & \vdots & & \vdots & \vdots & \ddots & \\
i_{|S|} & 0 & \cdots & p & \cdots & \cdots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

where the only non-zero entries are the values of p in the kth column in rows that contain elements of S.

We note here that several k-of bids for the same set S can be contained in the same matrix bid with "constant column" bids as illustrated in the previous example. For example, if we have the bids $\langle 2, \{a, b, c, d, e, f\}, p_2\rangle$, $\langle 4, \{a, b, c, d, e, f\}, p_4\rangle$, and $\langle 5, \{a, b, c, d, e, f\}, p_5\rangle$, we would form the matrix bid:

$$
\begin{array}{c|cccccc|}
a & 0 & & & & & \\
b & 0 & p_2 & & & & \\
c & 0 & p_2 & 0 & & & \\
d & 0 & p_2 & 0 & p_4 & & \\
e & 0 & p_2 & 0 & p_4 & p_5 & \\
f & 0 & p_2 & 0 & p_4 & p_5 & 0
\end{array}
$$

If at least two items are received the amount p_2 is promised to the auctioneer, if at least four items are received an additional amount p_4 is promised, etc. If instead the bidder wished to nest the exactly k-of bids $[2, \{a, b, c, d, e, f\}, p_2]$, $[4, \{a, b, c, d, e, f\}, p_4]$, and $[5, \{a, b, c, d, e, f\}, p_5]$, she may use a similar technique, making sure that the sum of the column values to the left of and including column k sum to the appropriate amount:

$$\begin{array}{c|cccccc|}
a & 0 & & & & & \\
b & 0 & p_2 & & & & \\
c & 0 & p_2 & 0 & & & \\
d & 0 & p_2 & 0 & p_4 - p_2 & & \\
e & 0 & p_2 & 0 & p_4 - p_2 & p_5 - p_4 & \\
f & 0 & p_2 & 0 & p_4 - p_2 & p_5 - p_4 & 0
\end{array}$$

In this case, if exactly four items are received then the second ranked item must have been priced at p_2 while the fourth ranked item must be priced at $p_4 - p_2$, summing to the amount p_4 as desired.

Just as several k-of bids can be nested in a single matrix bid, we now show that a chain of flat bids $(S_1, p_1), (S_2, p_2), \ldots (S_n, p_n)$ with $S_1 \subset S_2 \ldots \subset S_n$ can be nested along the diagonal as in the following matrix bid:

$$\begin{array}{c|ccccccc|}
S_1 & 0 & & & & & & \\
\downarrow & \vdots & p_1 & & & & & \\
S_2 & & & 0 & & & & \\
\downarrow & & & & p_2 - p_1 & & & \\
\vdots & & & & & \ddots & & \\
S_n & & & & & & 0 & \\
\downarrow & & & & & & & p_n - p_{n-1} \\
\vdots & \vdots & & & & & & \ddots
\end{array}$$

Here we have ordered the items so that all the items in S_1 are ranked higher than all the items in S_2, which are all ranked higher than the items in S_3, etc. We place the bid of p_1 for the set S_1 on the diagonal in the row of the last (lowest ranked) element in S_1, and a bid of $p_2 - p_1$ in the row of the last element in set S_2, so that receiving all elements in S_2 requires the acceptance of both bids p_1 and $p_2 - p_1$, summing to

p_2. Similarly, we put a bid of $p_l - p_{l-1}$ on the diagonal entry of the row with the last element of S_l. It is easily verified that this matrix bid yields the appropriate expression of preference, as assigning the last item in S_l causes the total accepted bid to reach p_l. We also notice that this bidder could assign bids for singleton sets in the first column without damaging the preferences already expressed there, making sure to reduce the diagonal entry containing $p_1$by the amount of the singleton bid on the item in the first row.

Because of its widespread use in simulations and general treatments of combinatorial auctions, it is interesting to consider the relationship of matrix bidding to an XOR-of-flat-bids language. In particular, we are interested in the following question: if a bidder can compute an optimal bid for any bundle and is given only a limited number of matrix bids to convey her preferences, is there a way to fill in matrix bid entries that makes safe (i.e. does not expose the bidder to paying too much for a bundle) but effective (i.e. is not dominated by another safe value) use of the space afforded by the matrix bid format?

Suppose, for example, that bidder j can quickly compute her desired bid on any bundle, $b_j(S)$, and is filling out a matrix bid to convey her preferences. She may first arrange the ranking of the auction items so that her most desired bundle is bid on exactly on the diagonal, and then proceed to consider how to fill in the other entries that are initially zero. Realizing that every bid on the diagonal and every bid in the first column are bids on exactly one subset (shown in the previous matrix bid example and the XOR of singletons example, respectively) she may simply fill these entries in. For example, in a four item auction we have the following matrix bid, where we abbreviate, for example, the set $\{A, B, C\}$ by ABC:

$$\begin{array}{c|cccc|} A & b_j(A) & & & \\ B & b_j(B) & b_j(AB) - b_j(A) & & \\ C & b_j(C) & x & b_j(ABC) - b_j(AB) & \\ D & b_j(D) & y & z & b_j(ABCD) - b_j(ABC) \end{array}$$

Our question from the previous paragraph now becomes: is there a way to choose entries x, y, and z that is safe and effective? The matrix bid paradigm makes simultaneous bids on several sets by assuming that the incremental value an item provides to a bundle does not vary over all bundles containing only higher ranked items. If in fact the incremental value an item provides to a bundle does vary over bundles containing only higher ranked items, the bid information in the interior entries (i.e. those not in the first column or on the diagonal) is not exact. These entries can be employed in a safe and effective way, however, by using the *minimum possible incremental value* for that entry. For this example we have:

$$x = \min(b_j(AC) - b_j(A), b_j(BC) - b_j(B))$$

$$y = \min(b_j(AD) - b_j(A), b_j(BD) - b_j(B), b_j(CD) - b_j(C))$$

$$z = \min(b_j(ABD) - b_j(AB), b_j(ACD) - b_j(AC), b_j(BCD) - b_j(BC))$$

Filling in the corresponding matrix bid entries with these values assures that the bidder has bid a safe amount, but could not have bid a higher safe amount. We note the interesting property that the number of terms in each minimization corresponds exactly to Pascal's triangle [**57**]:

$$\begin{array}{lllll} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \\ \vdots & & & & & \ddots \end{array}$$

This provides a characterization of the accuracy of each entry: b_{ijk} is an incremental value for item i if added to any of the $\binom{r_{ij}-1}{k-1}$ possible bundles containing $k-1$ more highly ranked items. This emphasizes that bid accuracy is exact in the left column and diagonal, and is least accurate in the middle of the bottom row. In general, this technique of finding a safe value for an arbitrary entry is given by:

$$b_{ijk}^{safe} = \min_{S \in SB(i,j,k)} b_j(S \cup i) - b_j(S)$$

where $SB(i,j,k) = \{S \subset I | r_{\bar{\imath}j} < r_{ij}, \;\; \forall \bar{\imath} \in S, \text{ and } |S| = k-1\}$

This "safe bid" technique allows a bidder to safely "pack in" more bid information into a matrix bid with a particular ranking or ordering of items already chosen. The method begins to answer the larger question of how to best use matrix bids to convey a general bid function, but to solve this larger problem the bidder must maximize bid accuracy over the $N!$ possible orderings, a problem that in general seems to require a great deal of enumeration. Consequently, the general bid function approximation problem for matrix bids is quite difficult due to the implied optimization over $N!$ different orderings. Future research on this subject may establish a direct comparison of matrix bidding to existing combinatorial auction benchmark data (e.g. the CATS data [**42**]). In the meantime, however, we continue our exploration of the types of preference information that can be compactly displayed in a matrix bid.

6.3.4. Contingency Expression Using $*$-entries

We now demonstrate another technique for preference expression with matrix bids that allows for a contingency relationship among "blocks" in the matrix. We will see that this allows for another form of "compactification" in which two or more "basic" expressions of preference can be combined in the same matrix bid, rather than spreading these basic pieces into separate matrix bids and joining them with logical connectives. Further, the ability to express this contingency relationship in a matrix bid allows us to omit a contingency operator from our logical language.

To demonstrate how to break a matrix bid into several pieces, consider that nothing in our formulation of MBA-IP or other descriptions of an MBA preclude the use of negative matrix bid entries. Indeed, though we do not study this subject here, it may be possible to conduct a combinatorial double auction using negative entries to represent a minimal payment request by the seller of the item. Because this requires added structure in the IP formulation, we maintain our original focus on a one-sided MBA, and save the study of a matrix bid double auction for future research.

Within the context of a one-sided auction, however, we do consider negative entries in one capacity: if a particular entry in a matrix bid is assigned a significantly negative number, then the revenue maximization will never assign the item in that row to the agent represented by that column. Representing this large negative number as $*$, we can block off regions of a matrix bid forcing entries to only be accepted in a certain way. We note that in practice we obtain the same results for $*$-entries by simply removing the decision variable associated with that entry from the IP formulation. Also, we notice that the use of $*$ as a prohibitive bid disallowing the acceptance of a certain bid is similar to the use of $*$ in our characterization of exactly-k-of bids in terms of k-of bids above.

To demonstrate the use of $*$, consider a bidder who wishes to express a contingency relationship between two "basic" valuations which can each be written in its own matrix bid. Suppose, for example, that the bidder is interested in purchasing a set of three items $\{A, B, C\}$ as a package at a price of 40, and would like to express an additive valuation over items $\{D, E, F\}$ contingent upon winning the essential set $\{A, B, C\}$. Using one block for the flat bid portion of these preferences and an additive block for the other, this bidder may place the following matrix bid:

$$\begin{array}{c|cccccc|}
A & 0 & & & & & \\
B & 0 & 0 & & & & \\
C & 0 & 0 & 40 & & & \\
D & * & * & * & 16 & & \\
E & * & * & * & 14 & 14 & \\
F & * & * & * & 12 & 12 & 12
\end{array}$$

Note, for example, that the set $\{B, C, D, E\}$ cannot be awarded to this bidder at a price of 14 (as it could if the $*$s were replaced with zeros) because in pricing this bundle a $*$-entry is accepted, representing a penalty great enough to overcome any generated benefit.

This ability to block-off "basic" expressions of preference quickly generalizes: any expressions that can be made in a separate matrix bid can be combined with a contingency relationship, as long as the subsets of items effectively bid on in each matrix bid is disjoint (where the items effectively bid on are those for which there is a positive entry in its row or below). A simple and powerful example of this is given by the following type of preferences which we call a *grocery-list bidding*. In this scenario a bidder perceives some list of necessary *ingredients* each of which must be obtained in a certain quantity and each of which has a list of substitute items.

For example, suppose the manager of a pizza company is purchasing ingredients at a wholesale auction (suppose perhaps that each item is a truckload of some ingredient). He may divide the set of auctioned items into categories of items, each containing several substitutable alternatives: doughs, sauces, cheeses, and toppings.

Figure 6.2. A grocery-list Matrix Bid

dough A	0								
dough B	0	0							
dough C	0	0	0						
dough D	0	0	0	*					
cheese A	*	*	*	0	*				
cheese B	*	*	*	0	0	*			
cheese C	*	*	*	0	0	*	*		
cheese D	*	*	*	0	0	*	*	*	
sauce A	*	*	*	*	*	40	*	*	*
sauce B	*	*	*	*	*	48	*	*	*
sauce C	*	*	*	*	*	52	*	*	*
topping A	*	*	*	*	*	*	7	*	*
topping B	*	*	*	*	*	*	6	6	*
topping C	*	*	*	*	*	*	5	5	5

Suppose further that his preferences are as follows. (i) He needs to obtain exactly three of the four types of dough, two of the four types of cheese, and one of the three types of sauce, or else he is not willing to purchase anything. (ii) Further, he perceives different quality levels for the various sauces, making him willing to pay 8 additional monetary units if his bundle contains sauce B, and 12 units if his bundle contains sauce C. (iii) Finally, he has additive preferences over three toppings, given that he has a bundle that satisfies his dough, cheese and sauce requirements. His preferences may be compactly expressed in the matrix bid Figure 6.2.

Applying the basic assumption for matrix bids that matrix bid entries may only be accepted in a down and to the right fashion, it is easy to see that the bid of Figure 6.2 expresses no positive bid except on bundles that contain exactly

Figure 6.3. Matrix Bids for spectrum licenses

Bid 1

SanJose	0		
Oakland	0	0	
San Fran.	0	0	50

Bid 2

Pasadena	0			
LongBeach	0	0		
Anaheim	0	0	0	
L.A.	30	35	45	60

three doughs, two cheeses, and one sauce, and that topping preferences are additive thereafter.

6.3.5. Using a Logical Language of Matrix Bids

Up to this point, this section has focused on expression of preferences within a single matrix bid. Matrix bidding is, however, a restrictive format and we emphasize that in practice a logical language of matrix bids would be required to convey preferences accurately. Though the previous section demonstrates a technique for joining basic preference expressions within a single matrix bid, many relationships can only be expressed by combining several matrix bids with logical connectives, as the following telecommunications example demonstrates.

Suppose we are auctioning city-wide licenses for mobile communication rights. One company may submit the two bids in Figure 6.3 for California cities. The metropolitan areas of $SanFrancisco$, $Oakland$ and $SanJose$ are close enough geographically that it would be impractical to purchase rights to one city without the other two; Bid 1 offers a single-minded bid of 50 for all three, but otherwise make no offers on any cities. Bid 2 introduces a different type of behavior. The bidding company is most concerned with purchasing the rights to $L.A.$, but feels that they will experience increased returns for each of the neighboring cities they obtain. Furthermore, they cannot support a communications network in the smaller neighboring cities without at least controlling the hub, $L.A.$ They bid 30 on $L.A.$ by itself, 35 on $L.A.$ with any one of its neighbors, 45 on $L.A.$ with any two of its neighbors, and 60 on $L.A.$ with all three of its neighbors. In general, if a bidder considers one item to be essential, with a set of n other items as substitutable accessories or add-ons, she may make bids on the distinguished item with 0 accessories, 1 accessory, . . . n accessories by bidding with an increasing row (we call such a bidder an *add-on bidder*).

If the two bids in this example are treated as if the single bidder had two bidders acting on his behalf, the submission is interpreted as "Bid 1 OR Bid 2". The OR is inclusive; the company submitting these bids may receive a subset which is the union of subsets obtained by each of its matrix bids. They may receive, for example, $\{SanJose, Oakland, SanFran.\} \cup \{L.A., LongBeach\}$ and pay $50 + 35 = 85$ units for the bundle. It may be the case, however, that the company is only interested in purchasing the rights for one of the two regions; they wish to bid "Bid 1 XOR Bid 2".

One way to quickly accomplish a XOR bid is to add a constraint on the slack variables into the integer program MBA-IP. Observe that if slack variable

$y_{j1} = 1$, bidder j's first agent is completely without an item. According to the ordering constraints if there is no item at bidder j's first agent there can be no item in bidder j's second agent, and so forth; $y_{j1} = 1$ therefore implies that bidder j receives no items. Since each bidder represents a single matrix bid we may achieve "Bid 1 XOR Bid 2" by adding the constraint:

$$y_{11} + y_{21} \geq 1$$

where $j = 1, 2$ refer to Bid 1 and Bid 2 respectively. For the remainder of this subsection we use the convention that each $j \in J$ refers to a single matrix bid, rather than a bidder in the auction.

In general the XOR of matrix bids $1, 2...m$ may be expressed by:

$$\sum_{j=1}^{m} y_{j1} \geq m - 1$$

If a bidder submits a request to XOR matrix bids $1, 2...m$, and a separate request to XOR matrix bids $(m+1), (m+2)...n$, these submissions do not mutually exclude one another. This allows each bidder's submission to easily take the general form:

$$(MatrixBid1\ XOR\ MatrixBid2...\ XOR\ MatrixBid\ M_1)\ OR$$

$$(MatrixBid\ M_1 + 1\ XOR\ ...MatrixBid\ M_2)\ OR...$$

$$(MatrixBid\ M_{Q-1} + 1\ XOR\ ...MatrixBid\ M_Q)$$

specifying an OR-of-XOR of matrix bids language, where $M_q = \sum_{l=1}^{q} m_l$, and each m_l equals the number of XORed matrix bids in the lth of the Q different ORed clauses.

Also studied in the auction literature are XOR-of-OR bidding languages, which include the format initially proposed for the FCC's Auction #31 [**28**]. This can be modeled in the matrix bid language as follows. Add a binary variable for each OR string, and define $\mathcal{Q} = \{1, 2, ...Q\}$ as the XORed set of clauses, each an ORed string of matrix bids. A bidder's complete preference submission will be of the form:

$$(MatrixBid1\ OR\ MatrixBid2...\ OR\ MatrixBid\ M_1)\ XOR$$

$$(MatrixBid\ M_1 + 1\ OR\ ...MatrixBid\ M_2)\ XOR...$$

$$(MatrixBid\ M_{Q-1} + 1\ OR\ ...MatrixBid\ M_Q)$$

where $M_q = \sum_{l=1}^{q} m_l$, and each m_l is now equal to the number of ORed matrix bids in the lth of the Q different XORed clauses. For each $q \in \mathcal{Q}$ define $g_q = 0$ if any matrix bid in the ORed string q receives items, $g_q = 1$ otherwise. Then add definitional constraints to MBA-IP for each string q:

$$g_q \leq y_{j1},\ \forall\ j \in q$$

and the XOR constraint:

$$\sum_{q=1}^{Q} g_q \geq Q - 1$$

The first constraint ensures that if $g_q = 1$, then none of the matrix bids j in string q receives items, while the second constraint ensures that among all Q strings at most one has one or more matrix bids receiving items.

If instead we are interested in expressing a premium p that will be paid only if some collection of statements each expressed in a matrix bid is satisfied, we can form constraints based on the slack variables for the *last* agent in each matrix bid that must be satisfied for the statement to be valid. For example, consider the communication license example and suppose we want to express that a premium p will be paid if the single-minded $Bid1$ is satisfied and at least two of the $L.A.$ neighbors in $Bid2$ are awarded. We can model this by introducing a new binary decision variable h_p which equals 1 whenever the AND of the two statements is satisfied, and is zero otherwise. We now add the constraints:

$$y_{13} + h_p \leq 1$$

$$y_{23} + h_p \leq 1$$

and an objective term $p \cdot h_p$. The value of the premium p is therefore only allowed to enter the objective function value when both agents receives items. The generalization of this technique to model a premium for the AND of more than two matrix bids follows similarly.

Though we provide here several examples of potentially useful logical languages of matrix bids, many other are possible. Indeed, any of the logical bidding structures put forth by Boutillier [**9**] may be adapted to create a more or less complex logical language of matrix bids. Our examples show that in certain cases a formulation may be expressed in terms of the slack variables for a single agent from each matrix bid, rather than on a large collection of x_{ijk} assignment variables. We note that the particular choice of an appropriate logical language may vary from application to application, and that one direction for future research is to empirically measure the trade-off between expressability and computational burden in the selection of a logical language.

CHAPTER 7

Computational Experiments for Matrix Bidding

After exploring the expressability of the matrix bids format as a compact language for conveying a wide variety of preference information and categorizing the winner determination problem for an MBA as $\mathcal{NP}$-hard, we are interested to see how efficiently we can solve instances of our formulation MBA-IP. In the early phases of our investigation, we began with some small examples generated on an ad-hoc basis, using our own imagination and economic intuition. Next we generated problems randomly, selecting each entry in each matrix bid from the same set of possible values with equal probability. Many of these instances were easily solved by their LP relaxation, while a few trends appeared in the more difficult cases which motivated the heuristics and data generation presented in this chapter.

In our examination of fractional LP-relaxation solutions, we investigated how to generate valid inequalities (cuts) that eliminate such solutions. We noticed that cover inequalities (commonly used for 0-1 Knapsack problems [**47**]) can be adapted and applied to our formulation MBA-IP. In the context of matrix bids the concept of a cover inequality can be described as follows. In an MBA each bidder has N ranked agents. If agent (j, k) receives an item then bidder j receives at least k items.

Naturally then, for two or more agents from different bidders, if the sum of the k indices of these agents is more than the number of items N, then one of these two agents must not receive an item. For example in an MBA with $N = 6$, bidder 1's 4th agent and bidder 2's 4th agent cannot both receive items. These conditions can be expressed as linear inequalities which are satisfied by all integer solutions to MBA-IP, but may be violated by fractional extreme points in its LP-relaxation. In §7.1.1 we develop a heuristic for the separation and application of these cover inequalities which can be used to eliminate fractional solutions when identified.

The second trend we noticed in out initial attempts at winner determination was that many LP-relaxations yielding a fractional solution often had the same objective function value as the optimal integer solution. Coupling this with the observation that the y_{jk} variables are always non-decreasing in the value of k for a given bidder j, we were motivated to develop an objective perturbation on the slack variables y_{jk} that favors integer solutions over fractional solutions in the presence of multiple optima. This objective perturbation technique is described in §7.1.2.

In our initial experiments it was uncommon to find instances in which the LP-relaxations yielded fractional solutions, and we decided that it would be beneficial to generate simulated auction data in a more sophisticated way. The reasoning for this was that the very "lumpy" preferences, those with significant positive synergies, were not very likely under a scheme in which each matrix bid entry was drawn with equal probability from the same support; totally random instances seemed too easy. Since we knew that computational difficulties in combinatorial auction winner determination arise exactly from this lumpiness, we decided instead to simulate bidders who would emulate "typical" behavior, including several lumpy type preferences like the "k-of" and flat bidding described in Chapter 6. In §7.2, we describe how this data was generated and present the results of several rounds of experimentation using CPLEX with and without the heuristics developed in §7.1 to solve the instances of MBA-IP for our simulated data.

7.1. Computational Procedure

In this section we describe various computational techniques used to solve instances of MBA-IP. §7.1.1 develops a way of recognizing and implementing a particular set of cover-like inequalities to strengthen the LP relaxation of MBA-IP. In §7.1.2 we demonstrate a method of introducing tiny weights to the objective function, to attempt to perturb a fractional solution towards an optimal integer solution.

7.1.1. Cover Inequalities

We propose a set of valid inequalities analogous to the well-known cover inequalities used for solving 0-1 Knapsack problems [**47**]. A *cover (for an MBA)* is a set of n destination agents, $\{(j_1, k_1), (j_2, k_2), ...(j_n, k_n)\}$, such that $j_m \neq j_l$ for any $m \neq l$, and $\sum_{l=1}^{n} k_l > N$. If in addition, for each $m \in \{1, 2, ...n\}$, $\sum_{l \neq m} k_l \leq N$, we say that the cover is *minimal.* The simultaneous occurrence of items at each of these agents implies a violation of supply restrictions, therefore at least one agent in the cover must not receive an item. In our IP formulation, we can write the following *cover inequality for* MBA-IP.:

$$\sum_{l=1}^{n} y_{j_l k_l} \geq 1$$

The following two-bidder, two-item auction demonstrates how a cover inequality may be used to eliminate certain fractional solutions:

Bidder 1		
item A	8	
item B	8	0

Bidder 2		
item A	2	
item B	0	3

The optimal allocation is to give *item A* to Bidder 2 (at value 2) and *item B* to Bidder 1 (at value 8), with total value of 10. The LP-relaxation of MBA-IP, however, yields a fractional solution with total value of 10.5. The fractional assignment is as follows:

$\frac{1}{2}$ of *item A* to Bidder 1's first agent at value $\frac{1}{2} \times 8 = 4$

$\frac{1}{2}$ of *item A* to Bidder 2's first agent at value $\frac{1}{2} \times 2 = 1$

$\frac{1}{2}$ of *item B* to Bidder 1's first agent at value $\frac{1}{2} \times 8 = 4$

$\frac{1}{2}$ of *item B* to Bidder 2's second agent at value $\frac{1}{2} \times 3 = 1\frac{1}{2}$

This undesirable fractional extreme point may be cut from the feasible polyhedron with the introduction of the appropriate cover inequality. Observe that the value of y_{jk}, the slack at bidder j's kth agent, can be computed as one minus the total amount of items received in bidder j's kth column. Thus we have $y_{11} = 0$, $y_{12} = 1$, $y_{21} = \frac{1}{2}$, and $y_{22} = \frac{1}{2}$. When $y_{11} = 0$, Bidder 1 has received at least one item. Since there are only two items being auctioned, this should mean that Bidder 2's second agent should be empty-handed (slack variable y_{22} should be one). Since it is not, we may form the cover $\{(1,1),(2,2)\}$, with inequality:

$$y_{11} + y_{22} \geq 1$$

which is violated by the given fractional solution, but not by any integer assignment.

Rather than add the entire set of cover inequalities to the IP formulation (an inefficient approach since there will be an exponential number of them), we would like to add these constraints only when we know they have been violated. We are thus interested in solving the *cover inequality separation problem for MBA-IP,* that is, to determine a violated cover inequality for a given fractional solution $\overline{y}$. This is equivalent to finding a binary vector $\gamma \in \{0,1\}^{MN}$ such that:

$$\sum_{k=1}^{N} \gamma_{jk} \leq 1, \qquad \forall j \in J \tag{7.1}$$

$$\sum_{j=1}^{M}\sum_{k=1}^{N} k \cdot \gamma_{jk} \geq N + 1 \tag{7.2}$$

$$\sum_{j=1}^{M}\sum_{k=1}^{N} \overline{y}_{jk} \cdot \gamma_{jk} < 1 \tag{7.3}$$

Here each $\overline{y}_{jk}$ is the value of y_{jk} in the given solution. The cover selection variable $\gamma_{jk} = 1$ when agent (j,k) is selected into the cover, with $\gamma_{jk} = 0$, otherwise.

Constraints (7.1) dictate that at most one agent is selected from each bidder, while constraints (7.2) ensure that the selected agents indeed have k indices that sum to strictly more than the number of auctioned items N (i.e. we have selected a cover). Constraint (7.3) guarantees that the selected cover is in fact violated by the current solution $\overline{y}$.

In this section, we demonstrate a necessary condition for the existence of a solution to this problem and introduce a heuristic technique for finding cover inequalities (in lieu of solving this cover separation IP exactly). We also note that because the cover inequalities defined here differ from the standard cover inequalities, we may not apply typical separation heuristics.

We begin with the following observations:

Lemma 7.1. For any solution to the LP relaxation of MBA-IP,

$$\sum_{j=1}^{M}\sum_{k=1}^{N} y_{jk} = N(M-1)$$

Proof. Summing each of the N constraints from (6.1) (introduced in §6.2) yields

$$\sum_{j=1}^{M}\sum_{k=1}^{N}\sum_{i\in I|r_{ij}\geq k} x_{ijk} = N$$

while summing each of the MN constraints in (6.2) yields

$$\sum_{j=1}^{M}\sum_{k=1}^{N}\sum_{i\in I|r_{ij}\geq k} x_{ijk} + \sum_{j=1}^{M}\sum_{k=1}^{N} y_{jk} = MN$$

with the desired result following by subtraction. □

Lemma 7.2. For a given j, values of y_{jk} are non-decreasing in k.

Proof. Observe that

$$\sum_{\{i\in I|r_{ij}\geq k\}} x_{ijk} \leq \sum_{\{i\in I|r_{ij}\geq k-1\}} x_{ijk-1}$$

Together with

$$\sum_{\{i\in I|r_{ij}\geq k\}} x_{ijk} + y_{jk} = \sum_{\{i\in I|r_{ij}\geq k-1\}} x_{ijk-1} + y_{jk-1} = 1$$

from (6.2), this yields $y_{jk-1} \leq y_{jk}$ as desired. □

One may observe these properties in the following table of $\overline{y}_{jk}$ values from a 12 item, 4 bidder LP relaxation of MBA-IP. Each row represents a bidder, j, with a column for each k value.

$$\begin{array}{r} \textit{Bidder } 1 \\ \textit{Bidder } 2 \\ \textit{Bidder } 3 \\ \textit{Bidder } 4 \end{array} \begin{bmatrix} 0 & 0 & 0 & \frac{1}{5} & \frac{1}{2} & \frac{4}{5} & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The cover separation problem is to pick entries, at most one from each row, such that the sum of the entries is strictly less than one, while the sum of the column indices is strictly greater than the number of items N (or equivalently $\geq N+1$, since column indices are whole numbers).

Can we find such entries for this table of $\overline{y}_{jk}$ values? First notice, that we can take the rightmost zero in each row, achieving a high value of k with zero contribution to the sum of $\overline{y}_{jk}$ weights. Here we get $3+1+2+2=8$ units towards the target of

13 (one more than N). As long there are some fractional values present, using these rightmost zeros will never be enough to create a cover (since the sum of all $(1-\overline{y}_{jk})$ values equals N and part of this sum lies in the fractional columns). Still, we have simplified our problem now to finding fractional values whose improvements to the cover total (i.e. the number of columns to the right of a rightmost zero) sum to at least $13-8=5$, while the sum of the selected entries sums to less than one. It is now easy to verify by inspection that there is no way of selecting fractional values from the table above, at most one per row, such that the values sum to less than one, while the sum of the column distance each fractional value is away from the last zero in its row is five or more.

The previous example provides some clues on how to find a violated cover inequality. To develop a general procedure, we compute a rating ρ_{jk} for each fractional $\overline{y}_{jk}$, indicating how strong a candidate it is for entry into the cover:

$$\rho_{jk} = \frac{k-k_j^*}{N+1-Z} - \overline{y}_{jk}$$

where k_j^* is the highest value of k for which $\overline{y}_{jk}=0$, and Z is $\sum_{j=1}^{M} k_j^*$, or equivalently, the total number of $\overline{y}_{jk}$ values equal to zero. When a row contains no zeros, we define $k_j^*=0$.

By dividing $k-k_j^*$, the number of columns away from a zero, by the amount of contribution needed from the fractional values, $N+1-Z$, we have normalized the problem so that a value's simultaneous impact on both constraints can be measured.

Thus if $\rho_{jk} > 0$, $\overline{y}_{jk}$ contributes proportionally more towards achieving the column sum (of $N+1-Z$) than it adds to the total fractional weight, which must remain below one. If $\rho_{jk} < 0$, then $\overline{y}_{jk}$ contributes proportionally more to the fractional weight than it does to achieving the desired column sum and is less likely to be an appropriate choice.

We now show that if $\rho_{jk} \leq 0$ for all (j,k) pairs, then no cover inequality is violated.

Lemma 7.3. If a violated cover inequality exists then there exists a (j,k) pair in the cover with $\rho_{jk} > 0$.

Proof. Suppose there is a violated cover inequality with $\rho_{jk} \leq 0$ for all (j,k) in the cover. Define $\overline{\gamma}_{jk} = 1$ if (j,k) is in the cover, 0 otherwise. Starting from the supposition that $\rho_{jk} \leq 0$, we have:

$$\rho_{jk} \leq 0 \Longrightarrow \sum_{j=1}^{M}\sum_{k=1}^{N}\overline{\gamma}_{jk} \cdot \rho_{jk} = \sum_{j=1}^{M}\sum_{k=1}^{N}\overline{\gamma}_{jk}\frac{k-k_j^*}{N+1-Z} - \sum_{j=1}^{M}\sum_{k=1}^{N}\overline{\gamma}_{jk}\overline{y}_{jk} \leq 0 \Longrightarrow$$

$$\frac{1}{N+1-Z}\sum_{j=1}^{M}\sum_{k=1}^{N}\overline{\gamma}_{jk}(k-k_j^*) \leq \sum_{j=1}^{M}\sum_{k=1}^{N}\overline{\gamma}_{jk}\overline{y}_{jk} < 1$$

with the last inequality holding because the specified cover is assumed to be violated. Then

$$\frac{1}{N+1-Z}\sum_{j=1}^{M}\sum_{k=1}^{N}\overline{\gamma}_{jk}(k-k_j^*) < 1 \Longrightarrow \sum_{j=1}^{M}\sum_{k=1}^{N}\overline{\gamma}_{jk}(k-k_j^*) < N+1-Z$$

Notice that $k_j^* = 0$ for a row with no zeros, and that we may safely assume that every bidder j with $\bar{y}_{jk} = 0$ for some k is in the cover, because if not we may safely add bidder j to the cover without harm to the constraints (7.2) or (7.3). Consequently,

$$\sum_{j=1}^{M}\sum_{k=1}^{N}\overline{\gamma}_{jk}k_j^* = Z$$

yielding,

$$\sum_{j=1}^{M}\sum_{k=1}^{N} k \cdot \overline{\gamma}_{jk} - \sum_{j=1}^{M}\sum_{k=1}^{N}\overline{\gamma}_{jk}k_j^* < N + 1 - Z \Longrightarrow \sum_{j=1}^{M}\sum_{k=1}^{N} k \cdot \overline{\gamma}_{jk} < N + 1$$

But with an insufficient column sum, the collection specified by $\overline{\gamma}_{jk}$ is not a cover, a contradiction. □

When some ρ_{jk} are positive, a violated cover inequality may or may not exist. When one does exist it may use a (j, k) pair with negative ρ_{jk}, and necessarily must use at least one of the (j, k) pairs with ρ_{jk} positive, as the following example demonstrates:

$$\overline{y}_{jk} = \begin{bmatrix} 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{1}{4} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{1}{4} & \frac{1}{2} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{1}{5} & \frac{1}{5} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\rho_{jk} = \begin{bmatrix} \cdot & -\frac{2}{35} & \frac{3}{35} & \frac{8}{35} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -\frac{3}{28} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -\frac{3}{28} & -\frac{3}{14} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -\frac{2}{35} & \frac{3}{35} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

In this example, two violated covers exist, both using a (j,k) position with positive ρ_{jk} and one with negative ρ_{jk}. One of these is specified by the violated minimal cover inequality: $y_{1,4} + y_{2,2} + y_{3,2} + y_{4,3} \geq 1$.

Armed with Lemma 7.3, we now develop the following Cover Separation Heuristic, summarized by the following description, with more a precise pseudocode description given in Algorithm 7.4 below.

We first add to the cover any (j,k) positions that add no fractional burden (because $\overline{y}_{jk} = 0$) and cannot be improved upon (since the next value $\overline{y}_{j(k+1)} = 1$). It is convenient to do this first because we may easily compute Z and k_j^* (needed for each ρ_{jk} calculation) along the way. Next, we compute ρ_{jk} values, checking to see if all s_{jk} values are non-positive, which would imply that no violated cover exists (by Lemma 7.3) allowing the heuristic to terminate. Otherwise, we look for promising, that is most positive, values of ρ_{jk} and add the corresponding (j,k) as long as the contribution to the fractional weight does not cause a violation of constraint (7.3). After all positive values have been tried, we attempt to add less promising (j,k) with non-positive ρ_{jk} to the cover, before finally settling on using the rightmost zero in a row with some fractional value(s) of $\overline{y}_{jk}$. We never attempt to use any but the best fractional candidate in a given row (in terms of ρ_{jk}) and maintain feasibility with regards to the bound on fractional weight throughout. Formally, we now present the procedure as used in our computational experiments.

Algorithm 7.4. Cover Separation Heuristic

(1) Initialize $COVERWEIGHT = FRACTIONWEIGHT = 0$

(2) For $j = 1$ to M:

 Determine position of rightmost $y_{jk} = 0$ and leftmost $y_{jk} = 1$

 If these are adjacent (i.e. $y_{jk} = 0$ and $y_{j(k+1)} = 1$):

  add (j, k) to the cover set

  update $COVERWEIGHT = COVERWEIGHT + k$

(3) For each fractional value of y_{jk}, compute ρ_{jk}

(4) If $\nexists \rho_{jk} > 0$:

  Terminate: no violated cover exists.

(5) For $j = 1$ to M:

 If $\nexists k$ with (j, k) in the cover set:

  Find the (j, k) pair with the highest value of ρ_{jk}

  If $\rho_{jk} > 0$ and $FRACTIONWEIGHT + y_{jk} < 1$:

   Add (j, k) to the cover set.

   update $COVERWEIGHT = COVERWEIGHT + k$,

$FRACTIONWEIGHT = FRACTIONWEIGHT + y_{jk}$

If $COVERWEIGHT \geq N + 1$:

Terminate: a violated cover has been found.

(6) For $j = 1$ to M:

If $\nexists k$ with (j, k) in the cover set:

Find the (j, k) pair with the highest value of ρ_{jk}

If $FRACTIONWEIGHT + y_{jk} < 1$:

Add (j, k) to the cover set

update $COVERWEIGHT = COVERWEIGHT + k$,

$FRACTIONWEIGHT = FRACTIONWEIGHT + y_{jk}$

If $COVERWEIGHT \geq N + 1$:

Terminate: a violated cover has been found.

(7) For $j = 1$ to M:

If $\nexists k$ with (j, k) in the cover set:

Add (j, k_j^*) to the cover set

update $COVERWEIGHT = COVERWEIGHT + k_j^*$

If $COVERWEIGHT \geq N + 1$:

Terminate: a violated cover has been found.

(8) $COVERWEIGHT < N + 1$

Terminate: the heuristic has failed to find a violated cover.

7.1.2. Objective Perturbation

After finding a solution to the LP-relaxation of MBA-IP and re-optimizing after the addition of violated cover inequalities, the solution may still not correspond to an integer allocation. In early experiments, we often found that the LP-relaxation objective value was equal to the optimal integer objective function value; in many cases there may be several solutions, both integer and fractional, on the optimal facet of the feasible polyhedron. When branch-and-bound is used to iteratively solve LPs, finding an integer extreme point among the numerous fractional extreme points on the optimal facet can be very time consuming. In this section we suggest a technique for making tiny shifts (or perturbations) to the objective function in order to favor the integer extreme points among all optimal solutions to an LP-relaxation (possibly after the addition of cover inequalities).

Recall from the discussion of cover inequalities that many fractional solutions have fractional slack variables $\overline{y}_{jk}$, while every integer solution must have integer $\overline{y}$s. Recall also that for a given j, values of $\overline{y}_{jk}$ are non-decreasing in k. Suppose we place a tiny negative weight on each slack variable that is increasingly negative for lower values of k. If we re-solve the LP relaxation, an integer solution with slack on the lower values of k is favored over a fractional solution which has some slack on $\overline{y}_{jk}$s with higher values of k. This tiny penalty on higher values of k allows us to favor integer solutions over fractional solutions that have the same objective value without the penalty. At times this objective perturbation is enough to achieve the optimal integer solution; in some instances it merely provides a better starting basis for branch-and-bound (sometimes a drastically better basis).

Given a particular LP-relaxation solution with associated $\overline{y}_{jk}$ values, we add objective function coefficient δ_{jk} for each y_{jk}, where:

$$\delta_{jk} = \frac{(\overline{y}_{jk} - 1)}{2N}$$

If $\overline{y}_{jk} = 0$, y_{jk} receives the most negative weight and is thus discouraged from being set to anything other than zero. When $\overline{y}_{jk} = 1$ the weight is zero, and there is no pressure to set $\overline{y}_{jk}$ to zero. In between, larger fractional values of $\overline{y}_{jk}$ incur less negative weight than small values, achieving the desired effect.

This procedure may be performed iteratively; after each LP-solution is obtained recompute the δ_{jk}s and re-optimize. What happens if an integer solution is found? The next calculation of perturbations will produce $\delta_{jk} = -\frac{1}{2N}$ wherever $\overline{y}_{jk} = 0$, and $\delta_{jk} = 0$ wherever $\overline{y}_{jk} = 1$. Since this will not change the objective value of the previous integer solution, the ensuing LP optimization will return the same solution. It is therefore safe to take a repeated solution as a stopping criterion for iterative perturbation; you cannot perturb to an integer solution and then away from it.

If the iterative perturbation procedure arrives at an integer solution, can we guarantee that it is an optimal integer solution to the unperturbed problem? We know that if we arrive at an integer solution we will settle there, but could we accidentally jump past an *optimal* integer solution and settle on an inferior one? To see that the answer is *no*, denote the unperturbed MBA-IP objective function value for specific variable values (x, y) as $obj(x, y)$, and the perturbed value for a specific vector of weights δ as $obj_\delta(x, y)$. We propose the following bound on the effect of perturbation on the objective function, with the desired result following as a corollary:

Lemma 7.5. $obj(x, y) - obj_\delta(x, y) < \frac{1}{2}$

Proof.

$$\sum_{j=1}^{M}\sum_{k=1}^{N}(-\delta_{jk}) = \sum_{j=1}^{M}\sum_{k=1}^{N}\frac{(1-\overline{y}_{jk})}{2N} = \frac{1}{2N}\sum_{j=1}^{M}\sum_{k=1}^{N}(1-\overline{y}_{jk}) = \frac{1}{2N}(NM - \sum_{j=1}^{M}\sum_{k=1}^{N}\overline{y}_{jk}) = \frac{1}{2}$$

With the last equality following from Lemma 7.1. Now, since not all y_{jk}s cannot simultaneously equal one, we have:

$$obj(x, y) - obj_\delta(x, y) < obj(x, y) - (obj(x, y) + \sum_{j=1}^{M}\sum_{k=1}^{N}\delta_{jk}) = \frac{1}{2}$$

□

Corollary 7.6. If the perturbation procedure terminates with integer variable values $(\overline{x}, \overline{y})$, then $(\overline{x}, \overline{y})$ is an optimal integer solution of MBA-IP.

Proof. Suppose not. Let $(\overline{x}_0, \overline{y}_0)$ be an optimal integer solution to MBA-IP. Since $(\overline{x}, \overline{y})$ is not an optimal integer solution to MBA-IP by assumption, $obj(\overline{x}_0, \overline{y}_0) > obj(\overline{x}, \overline{y})$, or equivalently $obj(\overline{x}_0, \overline{y}_0) - 1 \geq obj(\overline{x}, \overline{y})$ by integrality of the variables and objective function coefficients. Lemma 7.5 provides $obj_\delta(\overline{x}_0, \overline{y}_0) > obj(\overline{x}_0, \overline{y}_0) - \frac{1}{2} > obj(\overline{x}_0, \overline{y}_0) - 1$, which transitively tells us that $obj_\delta(\overline{x}_0, \overline{y}_0) > obj(\overline{x}, \overline{y}) \geq obj_\delta(\overline{x}, \overline{y})$, the last inequality following because all perturbation weights are non-positive. But now this says $obj_\delta(\overline{x}_0, \overline{y}_0) > obj_\delta(\overline{x}, \overline{y})$, regardless of the choice of δ, thus the perturbation procedure cannot terminate with variable values $(\overline{x}, \overline{y})$, since $(\overline{x}_0, \overline{y}_0)$ yields a better objective value for any δ. Thus we have reached a contradiction. □

Our approach is to iteratively perturb the objective function until a basis is repeated (with a reasonable bound on the number of iterations). If the solution obtained is integer Corollary 3.2 tells us that we are satisfied with this solution, and the perturbation has assuredly served us well. When the solution obtained from perturbation still has fractional variable values, we simply return the perturbed objective coefficients to zero and initiate branch-and-bound, maintaining the current solution. Though this would seem to imply that the objective perturbation was unsuccessful, our experimental results indicate that the new solution can significantly reduce the amount of time spent in the final branch-and-bound phase of optimization.

7.2. Experiments with Simulated Data

We report on our computational experiments and briefly describe our method of data simulation. The purpose of these experiments is to explore the efficacy of our computational techniques, to demonstrate the strength of the MBA-IP formulation, and verify that we can indeed rapidly solve MBA instances. To achieve these goals, data must be generated randomly, as little empirical data is publicly available for actual combinatorial auctions. The type of data simulation used here is common in the combinatorial auction literature. Sandholm et al.[**65**] is a prominent example, while Boutillier [**9**] notes that the available benchmark data (e.g. the CATS test suite discussed in [**41**]) is not beneficial for the testing of new language paradigms. Therefore as in [**9**], we use random generation of "typical sentences" in our own language as a test-bed for an initial evaluation of our ability to solve winner determination problems stated in this language.

Since many available combinatorial auction techniques focus on flat atomic bids and preferences built as an aggregation of flat bids, the data generated to test these techniques is naturally comprised of randomly drawn bids attached to randomly drawn subsets or bundles of items. Since matrix bidding is an auction format in which bidders assign values to bundles in a more orderly fashion, it should seem natural that our method is to randomly simulate expected behavior patterns. Our simulations proceed under the assumption that each bidder is of a type randomly drawn from each of the following seven types, which we use in an ad-hoc fashion to simulate the presence of a variety of different bidders in the market.

- Additive Preference Bidder
- Single-Minded Bidder
- Nested Flat-Bid Bidder
- Nested k-of Bidder
- Partition Bidder
- Add-on Bidder
- Diminishing Returns Bidder

To maintain simplicity in our experiments, we restrict to the situation in which each bidder is allowed only a single matrix bid (e.g. there is no XORing of matrix bids.) The input for each set of experimental runs is N (the number of items), M (the number of matrix bids), H (a parameter bounding the highest bid per item), and A (the number of auctions to simulate with these parameters).

Using these parameters, the simulation first randomly selects a bid profile for each matrix bid from one of the seven types described below with equal probability. For each bid type a subroutine randomly selects a nonnegative integer for each of the appropriate matrix bid entries based on the value H. We now describe how this is achieved for each of the seven bidder types used in our experiments. In each case (except for the partition bidder), before the entries in a matrix bid are filled in, a ranking r_{ij} is chosen at random from among the $N!$ possibilities.

Additive Preference Bidder: For each item in the auction, an integer is chosen from the set $\{0, 1, 2, ..., H\}$ with equal probability. Every row is then filled in with the same number equal to value chosen for the item in that row.

Single-minded Bidder: Included to simulate very inflexible or lumpy preferences, this bidder places a single positive entry on the diagonal. A column is chosen at random with equal probability, and a seed for the entry is chosen from the set $\{1, 2, ..., H\}$ with equal probability. The seed is then multiplied by the column number to reflect the number of items it is effectively bidding on.

Nested Flat-Bid Bidder: For each entry on the diagonal, an integer is chosen from the set $\{-H, ..., H\}$ with equal probability. Next, every negative selection is set to equal zero; there is thus a $\frac{H}{2H+1}$ probability of a positive entry, and probability of $\frac{1}{H}$ for each positive value conditional upon the entry being positive. Finally each positive entry is multiplied by the number of zero entries preceding it along the diagonal, weighting each entry by the number of items it has effectively bid on.

Nested k-of Bidder: Begin with a nested flat-bid bidder and simply fill in each column with the entry chosen for the diagonal entry in that column. An example is given by Figure 7.1. If the numbers 6, 4, and 7 are drawn for the columns 3, 5, and 6, respectively, each column generated by multiplying the random number by the

Figure 7.1. A nested k-of Matrix Bid

D	0					
B	0	0				
C	0	0	18			
A	0	0	18	0		
F	0	0	18	0	8	
E	0	0	18	0	8	7

number of consecutive zero columns immediately to the left plus one. Each positive bid then has an average bid per item equal to the randomly drawn number for that column.

Partition Bidder: This bidder divides the set of items into g groups of substitutes and gives a price for receiving one item from the group, given that one item from each previous group has been received. First we choose the number of groups g with equal probability from 2 to $\frac{N}{2}+1$. Then each item is assigned to a group with equal probability. Next the rankings are chosen so that each item in an earlier group has an earlier ranking (ranking within a group will not matter). Then, the value for any item in the group is chosen from the set $\{-H, ..., H\}$ with equal probability. The value is set to zero if negative, and multiplied by the number of consecutive immediately preceding groups with value of zero. Again this is to ensure that each positive bid value reflects the number of items it is effectively bidding on. Finally, the matrix bid column corresponding to each group number is filled in with the value of the group for any row corresponding to an item in the group, and $*$ for every other row. The result is a special case of the grocery-list bidding from §6.3.4, in which only one item is demanded from each group of substitutes.

Add-on Bidder: This bidder bids on an essential item and some nonnegative amount for any number of add-on items. First the row of the item to be considered essential is chosen at random with equal probability. Next an initial value is chosen for the essential item which is placed in the first column of the row. For each following entry in the row an integer between zero and the initial value is chosen, and added to the previous value in the row until the row is filled.

Diminishing Returns Bidder: This bidder simulates weakly diminishing marginal returns for every item received, and thus is generated to have weakly decreasing rows and columns. The entry in the first row and first column (i.e., the value for the highest ranked item) is chosen from $\{0, 1, 2, ..., H\}$ with equal probability. Each entry in the first column is given a value between the value above it and half of that value, with a 50% probability of being equal to the entry above it and an equal probability of taking any integer value between if not equal. To start the next column once a column is filled, the diagonal entry from that column is generated in the same way, but adjusting down to the entry on its left if that number is smaller. The other entries in the new column are generated in the same way as the first column, based on the entry above, but now adjusted down to the value of the entry on its left if necessary to ensure weakly decreasing rows.

These descriptions show that matrix bids entries in our random experiments were chosen so that the incremental value an item brings to a bundle is never more than the value H, and that when a particular entry effectively bids on multiple items, the random seed used to price the incremental value of a single item is multiplied by the number of items effectively bid on so that the bid entry is not drastically small. These measures assure that the preferences conveyed by each bid entry is "in the right ballpark" relative to all other bids in the auction. Further, they provide the right notion of "lumpiness" that experience suggests is the cause of computational difficulty in combinatorial auction winner determination.

Throughout the experiments presented here we maintain $H = 10$, though our glimpses at performance for other values of H suggest similar results. A more elaborate model might incorporate a different value H_i for each item i (i.e. a different support for each item's value), though here we assume that the value any item brings to a matrix bid entry is drawn from the same support.

After a set of randomly generated matrix bids is produced, each according to a particular bidder-type, the associated instance of MBA-IP is solved using (i) CPLEX only, and (ii) CPLEX augmented with the cover inequality and objective perturbation procedures outlined in §7.1. In the second case, the cover separation heuristic is executed after the initial LP relaxation is solved and repeated until the heuristic is unable to find violated covers. The objective perturbation is implemented

Figure 7.2. Average run time for winner determination

	$M = 5$	10	25	50	75	100
$N = 5$	0.024	0.038	0.065	0.137	0.213	0.272
10	0.071	0.140	0.339	0.579	0.866	1.222
25	2.812	1.967	3.890	11.035	14.098	22.739
50	34.022	29.208	134.24	587.43	1192.9	2451.66

following the cover inequalities procedure, since it is clearly rewarding to tightly characterize the feasible region (i.e. reduce it as much as possible) before perturbing. If these methods do not produce an integer solution, we commence branch-and-bound to determine an integer optimal solution, using CPLEX's default branch-and-bound settings.

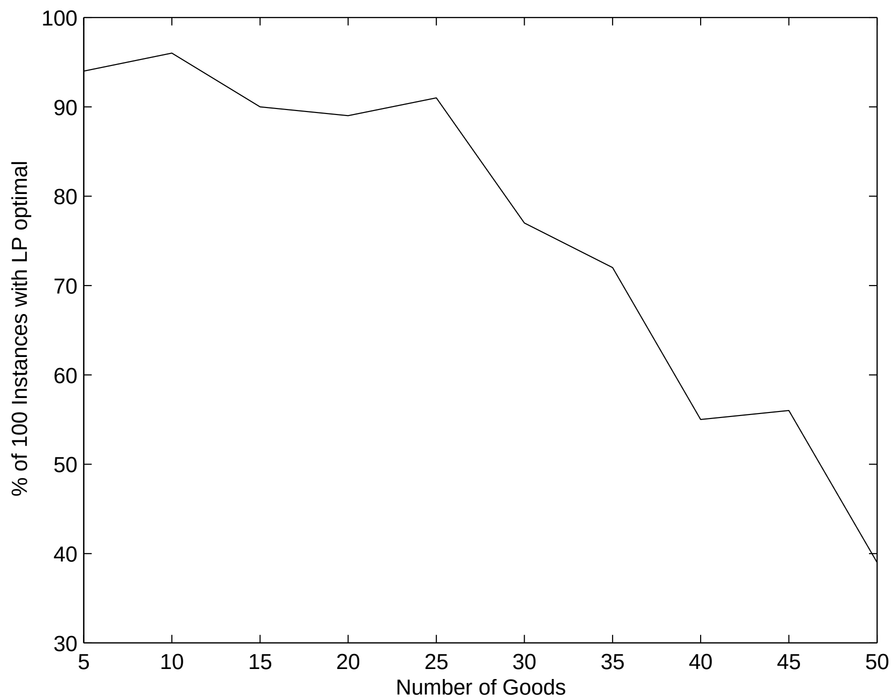
This procedure of solving the same winner determination problem two different ways is then repeated for A different simulated auctions. In the first round of experiments we set $A = 10$ for each value of (N, M) and examined the average, minimum, and maximum amount of computational time over the ten simulated auctions for each method. Only ten auctions were used at each point to quickly ascertain some global trends over a large number of points in (N, M) space. We repeated the procedure of ten auctions for every combination of N taken from $\{5, 10, 15, ...50\}$, and M taken from $\{5, 10, 15, ...100\}$ (since the size of the IP-formulation grows linearly in M and quadratically in N it seemed natural to be a bit more restrictive in our choices of N.)

Figure 7.2 provides a sample of the results, presenting only the average run times (in seconds) for CPLEX with our heuristics. (Throughout all testing, we use CPLEX version 8.0 on a Sun Microsystems Enterprise 250 with 2x400 MHz processors and 2GB RAM.) Figure 7.2 indicates that the winner determination integer program can be solved quickly enough for many practical applications. Multi-round auctions for several items often provide an hour or so per round to accommodate human deliberation, making run times of a few seconds for a small number of items and a few minutes for a modestly large number of items seem adequate.

Alternatively, if a sealed-bid auction is used in which VCG prices are computed through iterated solutions of the winner-determination problem, $P + 1$ IPs must be solved. One can quickly estimate that for a small number of items (even with a large number of matrix bids) the allocation and prices may be determined in under an hour. (Take for example ($N = 25, M = 100$) the 22 seconds per MBA-IP implies about 37 minutes for winner and price determination.) For a large number of items the complete price/allocation problem may take an entire day, but without the need for continual human interaction this may be acceptable.

As shown in Figure 7.3, a large portion of the LP-relaxations solve to integer optimality, especially for small number of items, a testament to the strength of the MBA-IP formulation. Because the initial LP-relaxation is strong to begin with, the heuristic techniques may not always be beneficial; CPLEX may have a relatively easy

Figure 7.3. Percentage of LPs solving to integral optimality vs. number of items, $M = 50$



time with MBA-IP as is. As such the heuristic techniques do not generally dominate the methods of CPLEX, nor does CPLEX always do better without the aid of these specially tailored algorithms. This behavior is often observed with difficult integer programs; rarely is one method "the best" for a wide class of problems. Since a state-of-the-art IP solver such as CPLEX employs such a large arsenal of techniques to more rapidly achieve optimal solutions, it is quite often satisfactory to improve CPLEX's performance *sometimes*, while never hindering performance too badly.

Figure 7.4. Worst case run time vs. number of matrix bids, $N = 50$

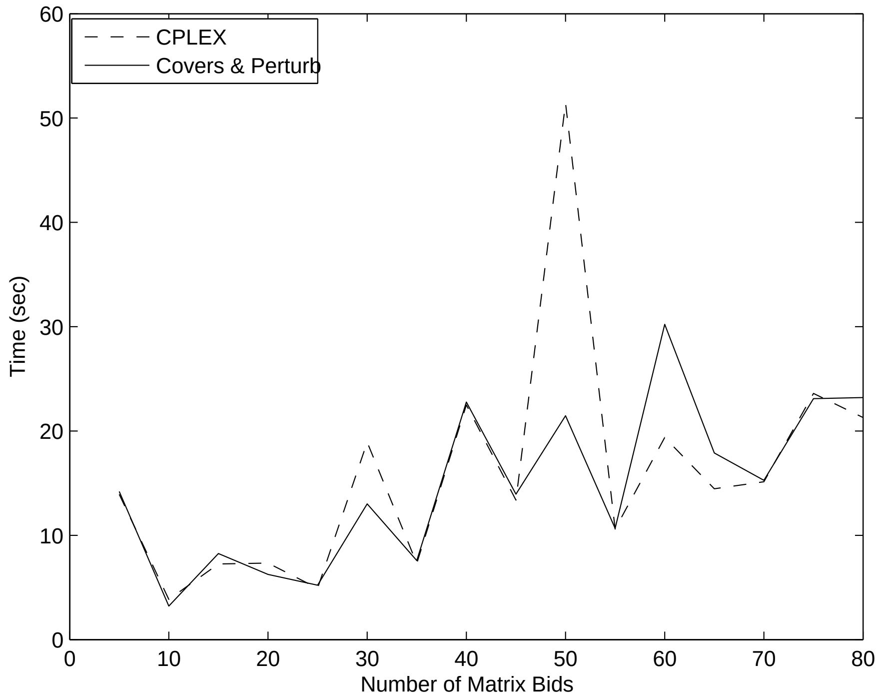
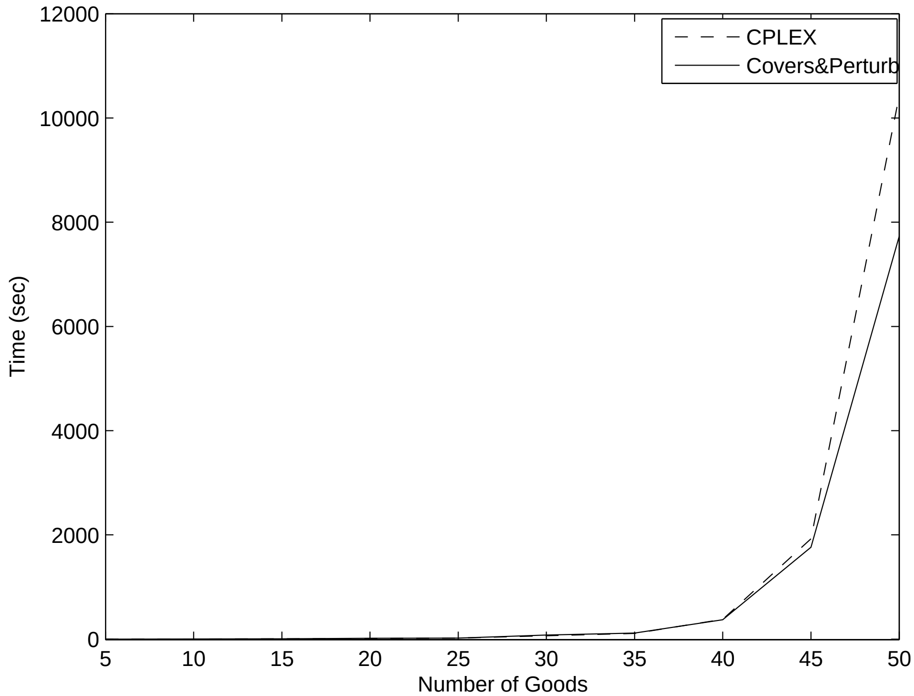


Figure 7.4 illustrates this sometimes improvement with a cross-section of worst-case run times holding the number of items constant at 25 (taken from the same data as above). The higher dotted peaks in this cross-section of data show that the addition of specialized cover inequalities and objective perturbation potentially diminish run time, while a solid peak above a dotted peak indicate that time spent on the heuristics may not always be advantageous. Typically, however, run times with and without the heuristics are similar.

Figure 7.5. Worst case run time vs. number of items, $M = 50$



This similarity, especially in average time, can again be attributed to the strength of the IP formulation and the high proportion of instances which are solved to optimality in the initial LP-relaxation phase. Figure 7.3 shows this proportion for our second set of data, in which the number of matrix bids M is held fixed at 50 while the number of items N is varied in steps of five from 5 to 50. For this new cross sectional data set we increased the parameter A to 100, so that each point in Figure 7.3 represents the percentage of 100 instances that were solved to optimality in the LP-relaxation phase.

This second set of data provides a more accurate comparison of worst-case run times while holding the number of bidders constant. We see in Figure 7.5 that the occasional differences between the two methods are smoothed out, and that in the worst case the two methods perform about the same. We do see the heuristics beginning to show more significant improvement for the largest instances, with a difference of about 45 minutes in the worst instance overall. Still, the apparent exponential growth in Figure 7.5 (typical of $\mathcal{NP}$-hard problems) suggests that in a multi-round setting MBA-IP would seem to be limited to auctions of 40 items or less, with the heuristics helping to shave at best 30% off the computational time of a run.

CHAPTER 8

Conclusions

Combinatorial auctions promise to increase efficiency and reduce exposure risk in an economic environment where "synergy" is significant. Applications for which combinatorial auctions achieve more desirable outcomes than traditional market mechanisms abound in both governmental allocation problems and B2B commerce, as discussed in §2.1. This increased satisfaction in market outcomes comes at the cost of computational difficulty, and the design of markets which quickly and easily achieve beneficial outcomes is certain to provide interesting new avenues of applied mathematics research for years to come.

With several potentially rewarding approaches to choose from (outlined in Chapter 2), this dissertation focuses primarily on the bidder's exponential bundles problem, asking: how can simple compact representations of preference (price-vectors) be combined to form more elaborate statements of preference? This question follows somewhat naturally from the economic literature of Kelso and Crawford [**33**], who introduce unit-demand bidders, each with preferences described by a price-vector, in a model of the job-market, in which an employee can accept at most one job. Kelso and Crawford [**33**] also introduce the gross substitutes property, which has become fundamental in the study of auctions. The strength of this concept in categorizing preferences and guaranteeing the existence of a unique Walrasian linear-price equilibria has influenced several authors, including Ausubel and Milgrom [**5**], and Gul and Stacchetti [**26**][**27**], all of whom provide foundational work for the research presented here. Other results on unit-demand bidders are given by Demange, Gale and Sotomayor [**19**], leading naturally into our own investigation of bid tables in Chapter 3. Indeed, the case of unit-demand bidders is well studied, and we note that our own Theorems 3.1 and 3.5 generalize the results of Demange, Gale and Sotomayor (as presented, for example, in §8.3 of Roth and Sotomayor [**60**]), from the case of unit-demand bidders to the case in which each bidder is represented by multiple unit-demand agents.

Representing bidders by multiple unit-demand agents results in the fairly natural and easy-to-read bid table format, causing us to wonder, why hasn't this been investigated before? We note that among the incentive properties of their unit-demand bidder auction model, Roth and Sotomayor [**60**] show that a bidder cannot benefit from shill bidding (having someone else join the auction to distort its outcome). This may seem to imply that representing a bidder by multiple unit-demand agents would be unrewarding, but the model under which this property was proven maintains the assumption of a unit-demand valuation function for each bidder. We, on the other hand, find use for the bid table format within the more general context of multi-unit demand.

How general is the preference expression afforded by the bid table format in the multi-unit demand context? Theorem 3.1 and Corollary 3.2 provide an exact characterization: assignment preferences (those which can be written in bid tables) are properly contained in the class of gross substitutes valuation functions. Applying a result of Gul and Stacchetti [**26**] based on this characterization, the gross substitutes property elucidates the greatest strength of a bid table auction: unique lowest Walrasian equilibrium price signals can be computed at each round of submission. The computation of these attractive linear price signals is facilitated by our constrained optimization approach, which allows us to recognize and neutralize "self-competition" constraints, overcoming the failure of the Hungarian algorithm to provide the truly lowest Walrasian equilibrium prices in this context, as it does in the unit-demand bidder context. With Theorems 3.4 and 3.5 we demonstrate the computation of these price signals in each round or static instance using the tools of constrained optimization, while Theorem 3.6 shows us that a Dynamic Bid Table Auction indeed converges to the desired outcome using lowest Walrasian prices as signals and a notion of straightforward bidding. Applying a result of Ausubel and

Milgrom [**5**], Theorem 3.3 establishes another strength of the static Bid Table Auction: it is a format for which the VCG mechanism is immune to the damaging effects of collusion, shill-bidding, and revenue reduction.

While the gross substitutes property does indeed provide several strengths of the bid table environment, it also clearly exposes its weaknesses. Most notably, preferences for complementary bundles can not be expressed in bid tables alone. We therefore introduced the Schedule Auction in Chapter 4 which implements a Dynamic Bid Table Auction as the first stage of a three-stage auction procedure. To design a second stage with demand revelation for complementary bundles, we examined the recent PAUSE auction of Kelly and Steinberg [**32**], making improvements based upon the use of bid tables in Stage I (to reduce negative synergy exposure) and a more precise understanding of the threshold and free-rider problems. Along the way, Theorem 4.1 demonstrated the $\mathcal{NP}$-completeness of the feasible allocation problem in a general demand correspondence reporting mechanism, justifying our decision to reject such formats (though used by Ausubel and Milgrom [**5**] and Gul and Stacchetti [**27**]) in favor of the adaptive user-selection approach of Kelly and Steinberg [**32**]. Lemma 4.2 verifies that this difficulty with demand correspondences does not appear when the gross substitutes property holds.

Motivated to develop a complete auction design for the airport landing slot application, for Stage III of our design we incorporated the state-of-the-art, bidder-Pareto-optimal payment mechanisms studied recently by Ausubel and Milgrom [**5**], Hoffman et al [**29**], and Parkes [**54**]. Our resulting algorithm allows us to compute these payments in any combinatorial auction that uses IP winner determination, with Theorem 5.2 establishing the validity of the approach. With Lemma 5.1, we refine the notion of a coalitional contribution function by computing each bidder's maximal coalitional contribution relative to the currently proposed payment vector. This recognition of each bidder's coalitional contribution and opportunity cost at the currently proposed payment vector allows us to iteratively generate violated core constraints at any payment vector. Our initial computations show that the resulting Core Constraint Generation technique compares favorably to other methods in the literature, and we look forward to further testing of this method.

With a complete auction format proposed in Chapters 1-5, Chapters 6 and 7 offer a new compact representation of preferences using price-vector agents in a more general environment. Bid tables are expressed as a standard two-dimensional array, giving them the valuable property of being able to be written in a spreadsheet or standard text editor, with only numbers and no need for special symbols or commands. A survey of the literature on bid languages shows that few if any have this property. Indeed, many use logical symbols that are not present on standard keyboards.

Given that expressability using bid tables is limited by the gross substitutes property, is there any other way to express a variety of preferences in a two-dimensional array, containing just numbers, that allows for expression of both substitutes and complements? In this dissertation we introduce matrix bidding and assert that it is such a format. Indeed, Theorem 6.1 shows that matrix bid expression is general enough for the winner determination problem to be $\mathcal{NP}$-hard, and thus fundamentally difficult. Despite this difficulty, the research of Padberg and Alevras [**52**] assures us that the problem of evaluating the bid on an arbitrary bundle can be performed in polynomial-time for any single matrix bid, with our own Lemma 6.2 providing a slight generalization of their result.

In §6.3 we investigated some of the strengths and weaknesses of matrix bidding in terms of preference expression. We emphasized that a logical language of matrix bids would be necessary for a reasonable generality of preference expression, and propose that an XOR-of-OR of matrix bid language admits a great variety of preference expression that can be written in a spreadsheet environment without the explicit use of connectives. The development and empirical testing of user-friendly interfaces (such as a spreadsheet implementation of an XOR-of-OR of matrix bid language) remain goals for future research, for which we have only taken a few initial steps.

In Chapter 7 we presented the first of these steps towards a matrix bid implementation. Faced with a new $\mathcal{NP}$-hard problem (the MBA winner determination problem), we showed how some of the general methods from the Operations Research tool kit (cover inequalities and objective perturbation) could be adapted for use in this new environment. The results of our simulated computational experiments showed that these techniques could sometimes be helpful, were never very harmful, and that reasonably sized problems could be handled relatively quickly by CPLEX with and without the aid of these heuristics.

Our method for data simulation discussed in §7.2 elucidates another potential strength of the price-vector agent formats. Matrix bidding and bid tables both provide quick ways to generate complex examples and simulated data using only numerical information. In the case of bid tables, Figure 3.3 provided a Bid Table Auction for which VCG payments were lower than the lowest Walrasian equilibrium payments, showing that bid tables are general enough to produce interesting examples, despite the limitations of the gross substitutes property. We propose that bid tables provide a way to generate gross substitute preferences that is much easier than the alternative, careful assignment of flat bids. Matrix bidding, on the other hand, provides a quick way to generate examples from a more general class of preferences.

As displayed in §7.2, however, some care must be taken that data is generated in an orderly fashion for the resulting matrix bids to represent economically meaningful behaviors.

In this dissertation, we have opened a new area of research for combinatorial auctions, focused primarily on the representation of preferences in two-dimensional arrays, interpreted as collections of price-vector agents. We hope to have convinced the reader that many of our techniques follow naturally from earlier literature, and that we have in some cases extended, applied, or generalized the work of others. As a work of *applied* mathematics, we hope to have convinced the reader that many of our techniques may be applicable to real world problems, providing more desirable market outcomes through scientific guidance, rather than remaining a mere mathematical curiosity. We hope that others will be influenced to join our pursuit of better economic solutions through the use of combinatorial auctions and other sophisticated market mechanisms.

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