OPTIONS AND EFFICIENCY *

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This paper argues that in an uncertain world options written on existing assets can improve efficiency by permitting an expansion of the contingencies that are covered by the market. The two major results obtained are, first, that complex contracts can be "built up" as portfolios of simple options and, second, that there exists a single portfolio of the assets, the efficient fund, on which all options can be written with no loss of efficiency.

An option contract is any security whose returns are dependent on the returns of some other underlying security (or securities). Warrants, puts, and calls are familiar financial examples of options written on stocks, but preferred stock and subordinated debentures and even such diverse contracts as life insurance policies could also be viewed as options. The serious study of options in the financial literature began with the long-neglected thesis by Bachelier and was revived in the 1960's by a number of authors who focused in on the pricing problem, i.e., the problem of determining the equilibrium relationship between the value of an option and the value of the stock it is written on.[1] The intention of this paper is to consider the related problem of the efficiency aspects of option contracts.

Arrow's[2] introduction of the state-space approach to uncertainty in economics has brought the recognition that an inadequate number of markets in contingent claims would be a source of inefficiency. In the state-space approach the random events that might occur are subsets of elementary points or "states" in a (probability) space, and the possibility of inefficiency arises whenever the feasible set of pure contingent claims, claims to wealth if a single state occurs and nothing otherwise, fails to span all the state space. An easy way to understand this is by analogy with a market where individuals are permitted to purchase a grapefruit only if they also buy an orange. If, by a fluke, everyone wishes to consume

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1. For a bibliography of this literature, see John C. Cox and Stephen A. Ross, "The Pricing of Options for Jump Processes," Rodney L. White Center for Financial Research, Paper No. 2–75, University of Pennsylvania.

2. Kenneth Arrow, "The Role of Securities in the Optimal Allocation of Risk-Bearing," *Review of Economic Studies,* XXXI (April 1964), 91–96.

one grapefruit with one orange, this constraint has no force. Otherwise, opening separate markets would improve efficiency.

This result, however, must be qualified. On the one hand, many of the states will be idiosyncratic to individuals, and events on these states will be independent across individuals permitting a simplification of the efficient market structure. Malinvaud [3] has recently confirmed this intuition and demonstrated that with large numbers of individuals a simple insurance program in lieu of the theoretically requisite complete contingency markets will remove this source of inefficiency. Second, economists have now begun to consider explicitly the impact of transactions and set-up costs of an institutional sort on equilibrium and efficiency.[4] If the introduction of a contingent claims market will use more resources than it will save, in an opportunity cost sense, by moving closer to efficiency, then within the context of the institutional structure of the economy the absence of the market is required for efficiency. Nevertheless, it is difficult to believe that such costs would be so prohibitive as to prevent the formation of nearly all contingent claims markets. Yet with the exception of some insurance examples, contingent contracts are difficult to find in actual markets. Even if we eliminate individualistic partitions of the state space, the number of states may greatly exceed the number of assets, and competitive equilibrium could be significantly inefficient.

The possibility of writing option contracts opens up new spanning opportunities. Although there are only a finite number of marketed capital assets, shares of stock, bonds, or as we shall call them "primitives," there is a virtual infinity of options or "derivative" assets that the primitives may generate. Furthermore, in general, it is less costly to market a derived asset generated by a primitive than to issue a new primitive, and there is at least some reason to believe that options will be created until the gains are outweighed by the set-up costs.

The main purpose of this paper will be to explore the relationship between primitive assets, derived options, and the attainment of theoretical efficiency. We shall not be concerned explicitly with transactions costs, but the relative cheapness of forming options, as opposed to primitives, underlies much of the analysis. The basic framework and definitions are presented in Section I and used to

3. E. Malinvaud, "Markets for an Exchange Economy with Individual Risks," *Econometrica*, XXXXI (May 1973), 383–410.

4. See, for example, F. H. Hahn, "Equilibrium with Transaction Costs," *Econometrica*, XXXIX (May 1971), 417–40.

prove a theorem relating general types of options and efficiency. In Section II some representation theorems among different types of options are obtained. The representation theorems are used to study the relationships between simple options and efficiency in Section III. In Section III we prove a rather surprising result that greatly simplifies the structure of option markets. In effect, we argue that there is a single efficient fund or portfolio of the primitive assets on which options can be written to enhance efficiency. Section IV summarizes and concludes the paper.

I

In the state-space framework, commodities are viewed as functions on the underlying state space. For simplicity and without loss of generality, we shall interpret each random vector as a security yielding returns denominated in dollars in each state of the world per unit investment. A typical asset x, then, is a map from the state space Ω to the line E:

$$x\colon\ \Omega \rightarrow E.$$

If the range of x is restricted to E^+, then the market is organized so that the asset offers limited liability. We shall assume that the state space Ω is finite, that $\Omega = \{\theta_1, \ldots, \theta_m\}$, and that there are n primitive assets $\{_1, \ldots, x_n\}$. The set of primitives is assumed to be invariant and cannot be altered, i.e., production decisions are precluded. We shall use X to denote both the mxn state-space tableau, with entries x_{ij}, the gross return on asset j in state i, and the set of n primitives.

Associated with X is the generated set,

$$P_X \equiv \{z \mid (\exists a \epsilon E^n) z = Xa\},$$

of derived assets attainable by forming portfolios a of the primitive assets. In the definition of P_X we permit short sales. We could, however, restrict z to be nonnegative to avoid the possibility of bankruptcy in any state. With this restriction P_X is a polyhedral cone in the positive orthant $E_+{}^n$, and the results below are unaltered.

If X has rank (or dimension) $\rho(X)$, then P_X lies in and spans a subspace of dimension $\rho(X)$. If $\rho(X) = m$, then there will exist a matrix of portfolios A, such that

$$XA = I_m,$$

i.e., X will possess a right inverse. This is equivalent to our being able to combine the primitives so as to form a complete set of pure contingent claims offering a return in only one state and zero in all the other states.

We shall assume that each of the states is critical in the economy in the sense that all of the states must be spanned (by contingent claims) to attain full Pareto efficiency. A sufficient condition for this to be true is that for each state there is some individual who values wealth in that state (and is not satiated). In an important sense, though, it is difficult to see how a state could appear in the tableau without its being critical for efficiency. States are merely elements in a minimal mathematical construct Ω, chosen to be just large enough to explain observed realizations. If a state appears in Ω, it is required to explain anticipated realizations, and as such must be critical. The criterion for efficiency then is that there exist assets to span all the states.

If, as is typical, there are more states than primitives, then we cannot span all of the states, and competitive equilibrium will be inefficient. Even though X fails to span Ω, however, it may be possible to augment the rank of X sufficiently by forming options on the existing primitives. This possibility is the focus of this paper. Of course, we are neglecting the consideration that the creation of markets in new assets will be costly. In general, efficiency must be assessed across alternative market and institutional structures. If costs are sufficiently high, it will be inefficient to open all the markets even if it does permit all the states to be spanned. (If costs are low, however, unless markets have significant public goods aspects, it is not clear why they will not be open in competition.) Our concern, though, will be solely with whether pure, or theoretical, efficiency is attainable. We shall use a crude ordinal notion of cost to establish a taxonomy of options. In effect, we shall prohibit some options as exorbitant in their resource use, and those that are allowed will be considered as costlessly marketable.

To begin with a concrete example, consider a call option written on an asset x. A call option promises a gross payment of

$$c_\theta(x;a) \equiv \max\{x(\theta)-a,\ 0\}$$

in state of the world θ, where a is the exercise or threshold price. Figure I illustrates the option contract. If

$$a \geqslant \max_\theta x(\theta),$$

then $c_\theta(x;a)=0$, and the call option will have a zero gross rate of return in all states. For

$$0<a< \max_\theta x(\theta),$$

the gross return will depend on the price of the option, which will be determined in the equilibrium. For all (positive) prices, though, the gross return will be proportional to $c_\theta(x;a)$, and this is all that we need to know for our purposes.

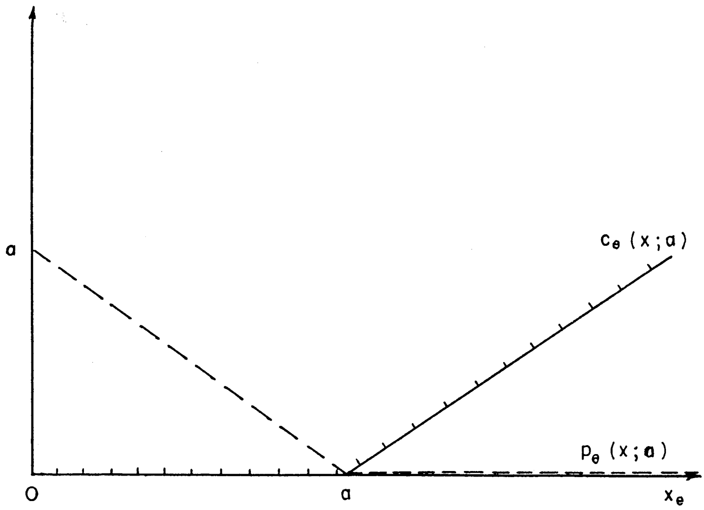

FIGURE I

If x has limited liability, then

$$c_\theta(x;0) = x(\theta),$$

i.e., a call with a zero exercise price is equivalent to the primitive asset on which it is written.

Similarly, we can define a put on an asset x by its gross payment,

$$p_\theta(x;a) \equiv \max\{0, a - x(\theta)\},$$

where a is the exercise price of the put. Inversely to a call, for a limited liability asset

$$p_\theta(x;0) = 0,$$

and in general, as first pointed out by Kruizenga,[5]

$$\begin{aligned} c_\theta(x;a) - p_\theta(x;a) \\ &= \max\{x(\theta) - a, 0\} - \max\{0, a - x(\theta)\} \\ &= x(\theta) - a. \end{aligned}$$

The following examples illustrate the use of options and some of their limitations in permitting the attainment of efficiency.

5. Richard J. Kruizenga, in *The Random Character of Stock Market Prices,* Paul Cootner, ed. (M.I.T. Press, 1964).

Example 1. Let X contain a single asset x with returns in the three states

$$x=\begin{bmatrix}1\\2\\3\end{bmatrix}.$$

By itself X cannot span $\Omega=\{\theta_1,\theta_2,\theta_3\}$, since $\rho(X)=1<3$. Forming calls on x with exercise prices 1 and 2, we have

$$c(x;1)=\begin{bmatrix}0\\1\\2\end{bmatrix}$$

and

$$c(x;2)=\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Now the rank of the augmented tableau,

$$[x \quad c(x;1) \quad c(x;2)]=\begin{bmatrix}1&0&0\\2&1&0\\3&2&1\end{bmatrix},$$

is full, and the call options permit us to attain efficiency.

Example 2. Let the single asset in X be

$$x=\begin{bmatrix}2\\2\\3\end{bmatrix}.$$

Now, all nontrivial call options on x have the form,

$$c(x;a)=\begin{cases}\begin{bmatrix}2-a\\2-a\\3-a\end{bmatrix} & \text{if } a\leqslant 2,\\ \begin{bmatrix}0\\0\\3-a\end{bmatrix} & \text{if } 2\leqslant a\leqslant 3.\end{cases}$$

Any augmented matrix formed with call options on x will have the first two rows identical and be of less than full rank. The same is true for put options as well.

The second example illustrates an important point. By definition an option is defined on the range of the random returns. The range defines the limited class of events that the random assets can distinguish among, and we cannot write options that distinguish between two states in which all assets have identical returns. (Presumably, though, the states are distinguishable from an efficiency viewpoint.) Quite generally, if we view the state tableau X as a mapping from Ω into E^n, i.e.,

$$X : \Omega \rightarrow E^n,$$

then a general or multiple option M is a mapping

$$M : E^n \rightarrow E,$$

giving a composite mapping $M(X(\theta))$ on states. It is important to emphasize that an option's return depends only on the return on the underlying assets it is written on and not on which state occurred. Letting M denote the class of general options and $O_X(M)$ the space spanned by X and all general options that can be written on X, we have the following simple result.

THEOREM 1. *The dimension of* $O_X(M)$ *is full if and only if no two rows of* X *are identical.*

Proof. If two rows θ_i and θ_j of X are identical, then for all multiple options

$$M(X_i) = M(X_j),$$

where X_k denotes the k^{th} row of X, and the augmented tableau $O_X(M)$ will not be of full rank. Conversely, if all rows of X differ, then we can define the option G_i as

$$G_i(X_j) = \begin{cases} 0 \text{ if } j \neq i \\ 1 \text{ if } j = i \end{cases}$$

and

$$[G_1(X_1)G_2(X_2) \; . \; . \; . \; . \; G_m(X_m)]$$
$$= I_m,$$

spanning all the states.

Q.E.D.

Theorem 1 is somewhat obvious, but it does serve to formalize the conjecture that a sufficient condition for spanning Ω is that for any two states there be some asset that distinguishes between them. It is clear that X is not necessarily of full rank simply because (for all i,j) $(\exists k)$

$$x_{ik} \neq x_{jk},$$

but when we augment X with multiple options, this condition is sufficient to permit us to distinguish among states and span Ω.

Multiple options, though, are really quite general and, in practice, contracts written on multiple contingencies are extremely rare. If we rule out such options as too costly, it still might be possible to augment the rank of X using simple options. By definition, a simple option O maps E into E, and if $x(\theta)$ is an asset, $0(x(\theta))$ is the return of the option in state θ. The class of simple options is thus quite large (essentially the class of all functions on E to E), but, fortunately, it is sufficient to consider only puts or calls.

II

Puts and calls are basic examples of simple options. In this section we shall show that all simple options can be thought of as portfolios of puts and calls. Before proceeding, however, there is one minor point to be taken up.

If there is some state θ, in which all assets give a zero return, then there are simply no resources available in this end of the world state. As such it is illusory to construct options that give a positive return in such states. We shall eliminate this problem by considering only the rank of the restricted tableau with the property that for each state there is some asset giving a positive return. This property will be referred to as the productivity assumption. We shall now prove in the next two theorems that it is sufficient to consider only call (or put) options to study the power of simple options.

THEOREM 2. *Let* $\mathrm{N_X}$ *denote the space spanned by all the simple options written on the primitives* X, *and let* $\mathrm{O_X}$ *denote the space spanned by the call (put) options that can be written on* X. *It follows that*

$$N_X = O_X.$$

Proof. Since calls are simple options, $O_X \subseteq N_X$. To prove the converse, let $y \epsilon N_X$. It follows that $(\exists \lambda_\gamma)$ and N^γ such that

$$(1) \qquad y = \sum_{\gamma=1}^{m} \lambda_\gamma N^\gamma,$$

where N^γ is a simple option written on some primitive asset.

It suffices, then, to show that each simple option is equal to a linear combination of puts and calls. Let x be a primitive asset on which one of the simple options N is written. Order x so that

$$x_1 \leqslant \ldots \leqslant x_m.$$

A basis for the space spanned by the calls on x is the set of calls

$$c^i \equiv c_\theta(x; x_{i-1}),$$

where we set $x_0 < x_1$.

Now partition the states into the k subsets $s_1, \ldots, s_k$, with k_i indices each, on which $x_i = x_j$ for $i,j \epsilon s_k$. By definition N is constant on each subset. Therefore, defining

(2)
$$\gamma_1 \equiv N_1/_{(x_1 - x_0)}$$
$$\gamma_i = 0 \text{ for } 1 < i \leqslant k_1$$
$$\gamma_{k_1+1} = \frac{N_{k_1+1} - \dfrac{N_1}{x_1 - x_0}(x_{k_1+1} - x_0)}{x_{k_1+1} - x_1}$$
$$\gamma_i = 0 \text{ for } k_1 < i \leqslant k_2,$$

etc., where N_i is the return on the simple option N in state i, we see that

$$N = \sum_{i=1}^{m} \gamma_i c^i.$$

The productivity assumption assures us that the calls are not illusory and the call options, alone, span N. A similar argument holds for put options.

Q.E.D.

If X is restricted to limited liability assets and we are not permitted to write calls with negative exercise prices, then Theorem 2 is no longer true. Consider the following example.

Example 3. Let

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Clearly X is productive, and $\rho(X) = 3$. Furthermore, all call options on the assets (columns) are simply proportional to the asset they are written on and, as such, they cannot augment the rank of x. Writing a put on the first asset, though, with a unit exercise price gives a return of

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix},$$

and

$$\left[\begin{array}{c|c} & 1 \\ X & 1 \\ & 1 \\ & 0 \end{array}\right]$$

is of full rank. The example is not unique, and with limited liability we can span the simple options by using both puts and calls.

THEOREM 3. *Let* N_X *denote the space spanned by all the simple options written on the primitives* X, *and let* O_X *denote the space spanned by the put and call options that can be written on* X. *It follows that*

$$N_X = O_X,$$

even if X *is limited liability and exercise prices are nonnegative.*

Proof. The proof is the same as that of Theorem 2 unless $x_1 = 0$, in which case we cannot write a call with an exercise price below x_1. By the productivity assumption, however, there must exist other assets $\{y^1, \ldots, y^{k_1}\}$ such that

$$y_i^i > 0.$$

This permits us to write a put on x with a positive exercise price a, less than x_{k_1+1}. Now, setting

$$\gamma_1 = \frac{N_1}{a}$$

in (2) and substituting the put for c^1 and removing $\frac{N_1}{x_1}$ in the formulas for γ_i, $i > 1$ permit us to span N as in (2).

Q.E.D.

III

In this section we prove our main result, Theorem 4, that there is a single portfolio or efficient fund of assets such that simple options on that portfolio will span the same space as general multiple options. Having obtained the representation theorems in Section II, we can study the class of simple options by studying puts and calls. We shall distinguish two situations. In the first case we are assumed to be able to write simple options not only on the primitives in X, but also on portfolios of the primitives, i.e., the elements of P_X. In the second case we assume that options can be written only on the primitives. As we shall see, the distinction is a meaningful one.

For any state-space tableau F, define O_F to be the space spanned by F, and all simple options that can be written on F. The first case is treated in the following theorem.

THEOREM 4. *A necessary and sufficient condition for* $\rho(O_{P_X}) = m$ *is that there exists a single portfolio* a *such that*

(3a) $\quad Xa = b$

with $b_i \neq b_j$ *all* (i,j).

If limited liability provisions apply, then we also require that

(3b) $\min_{\theta} b_\theta > 0,$

as a necessary and sufficient condition for efficiency.

Proof. Suppose that an $<a,b>$ pair exists. Since $b \epsilon P_X$, by Theorem 1 considering only options written on b we can span O_{P_X}. By Theorem 3 this can be accomplished with calls if (3b) is satisfied.

Conversely, suppose that there does not exist an $<a,b>$ pair satisfying condition (3a). This means that for any given a, $(\exists\theta,\theta')$

(4) $X_\theta a = X_{\theta'} a.$

The set

$$\sigma \equiv \{a \mid (\exists\theta,\theta' \neq \theta)\,(X_\theta - X_{\theta'})\,a = 0\}$$

is the union of a collection of linear manifolds,

$$\sigma = \bigcup_{\substack{\theta,\theta' \\ \theta \neq \theta'}} A_{\theta,\theta'},$$

where

$$A_{\theta,\theta'} \equiv \{a \mid (X_\theta - X_{\theta'})\,a = 0\}.$$

If we cannot obtain a solution to (3a) for any a, then $\sigma = E^n$. Since a finite union of linear manifolds cannot have a dimension in excess of the highest dimension among the component sets, $(\exists\theta,\theta')$

$$A_{\theta,\theta'} = E^n.$$

It follows that $X_\theta = X_{\theta'}$ and by Theorem 1, $\rho(O_{P_X}) < m$.

In the limited liability situation, if there does not exist an $<a,b>$ pair satisfying both (3a) and (3b), then for any given a $(\exists\theta,\theta')$ satisfying (4) or θ such that

$$X_\theta a \leqslant 0.$$

Defining

$$A_\theta \equiv \{a \mid X_\theta a \leqslant 0\},$$

we see that the set

$$\sigma U [U_\theta A_\theta] = {}^n.$$

If for some (θ,θ'), $A_{\theta,\theta'} = E^n$, the proof is as above. If not, then σ will not contain the positive orthant. Since X_θ is nonnegative, A_θ will be disjoint from the positive orthant, unless $X_\theta = 0$, violating the productivity assumption and, of course, not permitting us to span Ω with calls alone.

Q.E.D.

Theorem 4 is somewhat surprising. When we are permitted to write options on portfolios, a necessary as well as sufficient condition for efficiency is that there exists a single portfolio a with the property that options written on it can span Ω. This result permits us to link the simple options to the more general ones.

THEOREM 5. *The spaces* $O_X(M)$ *and* O_{PX} *are identical.*

Proof. Since a simple option written on a portfolio of assets in X is a multiple option on X, $O_{PX} \subseteq O_X(M)$. From Theorem 1, $O_X(M)$ is simply a subspace of E^m:

$$O_X(M) = \{x \mid x \epsilon E^m \text{ and } x_i = x_j \text{ if } X_{\theta_i} = X_{\theta_j}\}.$$

From the proof of Theorem 4, though, there exists a portfolio a with returns b such that $b_i = b_j$ if and only if $X_{\theta_i} = X_{\theta_i}$. For any $y \epsilon O_X(M)$, then, y is an arbitrary simple option on b and therefore $y \epsilon O_{PX}$.

Q.E.D.

In other words, by increasing the domain of simple options to include portfolios of primitive assets, we can span the same space with simple options as with multiple options. (Notice, of course, that

$$O_X(M) = O_{PX}(M),$$

so that nothing is gained by permitting multiple options to be written explicitly on portfolios.) This result is quite important; it may explain why so few multiple options are written in practice. If individuals are permitted to form portfolios, then there is a single portfolio a, such that any multiple option they might wish to write will be equivalent to a simple option on portfolio a.

Unfortunately, though, if we do not permit simple options to be written on portfolios, then the class of simple options is not as powerful as that of multiple options. Consider the following example.

Example 4. Let

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \\ 2 & 2 \end{bmatrix},$$

a productive tableau. Since each of its rows is unique, there exists a multiple option that spans Ω. Equivalently a portfolio with $a = (2,1)$ is equivalent to an asset with returns

$$2\begin{bmatrix}1\\1\\2\\2\end{bmatrix}+\begin{bmatrix}1\\2\\1\\2\end{bmatrix}=\begin{bmatrix}3\\4\\5\\6\end{bmatrix},$$

and calls written on the portfolio will also span Ω.

If we consider only simple options on X^1 and X^2, though, we cannot span Ω. By Theorem 2 it is sufficient to consider only call options. Augmenting X by the nontrivial calls on X^1 and X^2, we have

$$A\equiv[X\mid c^1\mid c^2]=\begin{bmatrix}1&1&0&0\\1&2&0&1\\2&1&1&0\\2&2&1&1\end{bmatrix}.$$

Since

$$A^1+A^4=A^2+A^3,$$

A is of less than full rank.

Our goal now is to characterize those X for which O_X is of full rank. To do this, we shall want to examine the space O_{X^i} for an individual asset X^i in X somewhat more closely. Let L_i be a matrix with rows that have all zeros except for a 1 and a -1 in positions k and l, where $X_{k^i}=X_{l^i}$. To illustrate, consider the asset

$$X^i=\begin{bmatrix}1\\2\\1\\3\\2\end{bmatrix}.$$

We can take

$$L_i=\begin{bmatrix}1&0&-1&0&0\\0&1&0&0&-1\end{bmatrix}.$$

If we change the 3 to a 1, we would add a row,

$$[0\ 0\ 1\ -1\ 0]$$

or

$$[1\ 0\ 0\ -1\ 0].$$

The order of the 1 and -1 in any row is irrelevant.

The importance of L_i is that it permits us to characterize O_{X^i}.

LEMMA 1.

$$O_{X^i}\equiv\{x\mid L_ix=0\}.$$

Proof. If $y\epsilon O_{X^i}$, then $y_k=y_l$ if $X_{k^i}=X_{l^i}$, which implies that

$L_i y=0$. Conversely, if $L_i y=0$, then $y_k=y_l$, whenever $X_k^i=X_l^i$. By Theorem 1, y is therefore an option on X^i.

Q.E.D.

Thus, the set of options is simply the space of vectors orthogonal to the rows of L_i. We can now characterize those X for which O_X is of full rank.

THEOREM 6. *The rank of* O_X *is full if and only if the row spaces of* $\{L_1, \ldots, L_n\}$ *are mutually orthogonal, i.e., there does not exist* $(a_1, \ldots, a_n)$ *with*

(5) $$a_1L_1=a_2L_2= \ldots =a_nL_n\neq 0.$$

Proof. Since we can only write options on the primitive assets,

(6) $$O_X=O_{X^1}+ \ldots +O_{X^n}.$$

Since O_X is the sum of linear spaces, it is a linear space itself. It follows that $\rho(O_X)<m$ if and only if $(\exists z\epsilon E^m z\neq 0)$ polar to O_X, i.e., (for all $y\epsilon O_X) zy=0$. From (6) (for all $y\epsilon O_{X^i}) zy=0$. Thus, z belongs to the polar set of O_{X^i},

(7) $$\begin{aligned} O^\dagger_{X^i} &\equiv \{z \mid (\text{for all } y\epsilon O_{X^i}) zy=0\} \\ &= \{z \mid (\exists a) z=aL_i\}, \end{aligned}$$

where we have used Lemma 1. From (7) it follows that $\rho(O_X)<m$ if and only if

$$\bigcap_{i=1}^{n} O^\dagger_{X^i}\neq\{0\},$$

which is equivalent to finding a $z\epsilon \bigcap_{i=1}^{n} O^\dagger_{X^i}$, $z\neq 0$, such that

$(\exists a_1, \ldots, a_n)$

$$z=a_1L_1= \ldots =a_nL_n.$$

Q.E.D.

Condition (5) is actually fairly straightforward to check in practice. By performing column operations, each $(kxm)L_i$ can be reduced to echelon form in k operations of the type "add column k to column l if there is a row θ with a 1 in column k and a -1 in column l." Condition (5) can now be verified by checking if any unit vector appears in all of the reduced L_i matrices.

Applying Theorem 6 to Example 4, we have

$$L_1=\begin{bmatrix}1 & -1 & 0 & 0\\ 0 & 0 & 1 & -1\end{bmatrix}$$

$$L_2=\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{bmatrix}.$$

Taking a portfolio with weight $+1$ on the first row and weight -1

on the second row, we obtain $(1 \ -1 \ -1 \ 1)$ for both L_1 and L_2, which means that the matrices are not mutually orthogonal. By Theorem 6 the rank of O_X is less than 4, as we verified earlier.

IV

This paper has studied the use of options to attain efficiency in competitive equilibrium in the absence of complete markets. Perhaps the most interesting characteristic of the results has been the finding that rather simple options have considerable power to accomplish this. Arbitrary simple options are equivalent to a portfolio of call options. Most important, though, complex multiple options are equivalent to simple options written on a portfolio of primitive, marketed assets. These reductions should act to simplify the use of options considerably, particularly in well-organized security markets.

Beyond this work at least two extensions seem clear. On the one hand, there is a need to strengthen the present results by weakening the mathematical structure. It should be possible to carry out all of the above analysis for rather arbitrary σ-algebras of events with appropriate equivalents on null sets. This would not appear to offer any real surprises, though. In an independent paper Schrems [6] has shown, for example, that if Ω is the line, then calls span simple options when the probability measure has a density representation.

Much more important will be the extension of these results to an intertemporal context. In such a context as the option pricing literature makes clear, we have to distinguish between types of simple options according to their characteristics over time. For example, it is well-known that an American put that may be exercised at any time before its expiration date is a quite different instrument than a European put that can be exercised only at the expiration date. The American call, though, is the same as the European call.[7] One conjecture suggested by this paper is that complex financial instruments are made up of portfolios of American puts and calls.

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6. Edward L. Schrems, "The Sufficiency of Existing Financial Instruments for Pareto-Optimal Risk Allocation," Ph.D. thesis, Stanford University, 1973.
7. See Robert C. Merton, "The Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science,* IV (Spring 1973), 141–83.